MATH 454 - Honours Analysis 3

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These notes are heavily based on Professor Jérôme Vétois's notes and lectures for MATH 454.

1 Lebesgue Measure

Start of chapter is review from MATH 356.

1.1 Outer and Inner Approximations of Measurable Sets

Theorem 1.1. Let $A \subseteq \mathbb{R}$. Then, the following are equivalent:

1. A is measurable
2. $\forall \epsilon > 0 : \exists O_\epsilon \subseteq \mathbb{R}$ open such that $A \subseteq O_\epsilon$ and $m^*(O_\epsilon - A) < \epsilon \iff m^*(O_\epsilon) - m(A) < \epsilon$
3. $\exists (O_m)_{m \in \mathbb{N}}$ open such that $G = \bigcap_{m \in \mathbb{N}} O_m$, $A \subseteq G$ and $m^*(G - A) = 0$
   - Such a set (countable intersection of open sets) is called a G_δ **set**.
4. $\forall \epsilon > 0 : \exists F_\epsilon \subseteq \mathbb{R}$ closed such that $F_\epsilon \subseteq A$ and $m^*(A - F_\epsilon) < \epsilon$.
5. $\exists (F_m)_{m \in \mathbb{N}}$ closed such that $F = \bigcup_{m \in \mathbb{N}} F_m$, $F \subseteq A$ and $m^*(A - F) = 0$
   - Such a set (countable union of clsoed sets) is called a F_σ **set**.

Proof. As always, proof by transitivity

1. $\implies$ 2. Assume A is measurable.

In the finite case $m(A) < \infty$, we have that $\exists (I_{k,\epsilon})_{k \in \mathbb{N}}$ open and bounded such that $\sum_{k=1}^{\infty} l(I_{k,\epsilon}) < m(A) + \epsilon$ and $A \subseteq \bigcup_{k=1}^{\infty} I_{k,\epsilon}$. Take O_ϵ to be the union of these intervals (also open as the union of open sets). Then by subadditivity:

$$m^*(O_\epsilon) \leq \sum_{k=1}^{\infty} m^*(I_{k,\epsilon}) \leq m(A) + \epsilon$$

Since $m(A) < \infty$, it follows by excision that $m(O_\epsilon - A) = m(O_\epsilon) - m(A) < \epsilon$.

For the case where A has infinite measure, break A into $A_k = A \cap [k, k+1)$ for all integers. Using the finite case above, we can find an open set such that $m^*(O_{k,\epsilon} - A_K) < \frac{\epsilon}{5} 2^{-|k|}$. Let O_ϵ be the union of these sets. Then

$$O_\epsilon - A = \bigcup_{k \in \mathbb{Z}} O_{k,\epsilon} - \bigcup_{k \in \mathbb{Z}} A_k \subseteq \bigcup_{k \in \mathbb{Z}} O_{\epsilon,k} - A_k$$

By monotonicity and subadditivity, we have that:

$$m^*(O_\epsilon - A) \leq \sum_{k \in \mathbb{Z}} m^*(O_{\epsilon,k} - A_k) < 4\epsilon + \epsilon$$

Where the last ϵ comes from $k = 0$

2. $\implies$ 3. Let $G = \bigcap_{i=1}^{\infty} O_{\frac{1}{n}}$ where $O_{\frac{1}{n}}$ is open such that A is contained in it and $m^*(O_{\frac{1}{n}} - A) < \frac{1}{n}$. Then, we have that $A \subseteq G$ and $m^*(G - A) \leq m^*(O_{\frac{1}{n}} - A) < \frac{1}{n}$ by monotonicity and the whole thing goes to 0 as $n \to \infty$.

3 $\implies$ 1. $A = G - v$ where $v = G - A$ and has measure 0 while G is G_δ. Both G_δ sets and sets of outer measure 0 are measurable, A is also measurable.

To show 4. and 5., notice that if A is measurable, so is its complement.

4. $\forall \epsilon > 0 : \exists O_\epsilon$ that satisfies 2. Then, considering the complement of O_ϵ gives us the required set.

5. Taking the complement of the family of open sets that form the G_δ set in 3. gives us the sets whose intersection fits the required properties. □

We now have 3 questions:

1. Do there exists non-measurable sets?
2. Do there exist uncountable sets of measure 0?
3. Can all measurable sets be obtained as some combination of complements and countable unions of open sets.

1.2 The existence of non-measurable sets

While we cannot explicitly construct a non-measurable set, we can prove its existence.

Def. Axiom of choice Let Ω be a collection of non-empty sets. Then $\exists f : \Omega \to \cup_{s \in \Omega}$ such that $\forall s \in \Omega : f(s) \in S$. Essentially, from an arbitrary collection of sets, we can always pick an element from each set.

Theorem 1.2. $\forall A \subseteq \mathbb{R}$, if $m^*(A) > 0$, then there is some $B \subseteq A$ that is non-measurable.

Proof. We begin by reducing the problem to bounded subsets of A

If it is unbounded, we can write:

$$A = \cup_{k \in \mathbb{Z}} A_k$$

where $A_k = A \cap [k, k+1)$

Then, by subadditivity, we have that:

$$0 < m^*(A) \leq \sum_{k \in \mathbb{Z}} m^*(A_K) \implies \exists k_0 \in \mathbb{Z} \text{ with } m^* A_{k_0} > 0$$

Then, taking that A_{k_0}, we can define:

$$\forall x \in A_{k_0}, S_x = (x + \mathbb{Q}) \cap A_{k_0}$$

To be an uncountable collection of countable sets. With Ω as the collection of all the S_x, applying the axiom of choice, we can define a function f that picks one element from each of the S_x. We seek to show that:

$$A_{k_0} \subseteq \bigcup_{q \in \mathbb{Q} \cap [-1,1]} f(\Omega) + q$$

Where $f(\Omega)$ is the set of all chosen points. Let $x \in A_{k_0}, f(S_x) \in S_x$. Then, to show it belongs in the RHS:

$$\exists q_x \in \mathbb{Q} : f(S_x) = x + q_x \text{ and } f(S_x) \in A_{k_0}$$

It follows that $x = f(S_x) - q_x$ and $|q| = |f(S_x) - x| \leq 1$ since $x, f(S_x) \in A_{k_0} \subseteq [k_0, k_0 + 1)$ and thus we can conclude that x is in the RHS.

Now, let $S_x, S_{x'} \in \Omega$ and $q, q' \in \mathbb{Q} \cap [-1, 1]$ be such that $f(S_x) + q = f(S_{x'}) + q'$. Then, since $f(S_x)$ is just $x + q_x, q_x \in \mathbb{Q}$, we have that:

$$x + q_x + q = x' + q_{x'} + q'$$
$$x + \mathbb{Q} = x' + \mathbb{Q}$$
$$S_x = S_{x'}$$

Then it must be that $f(S_x) = f(S_{x'}) \implies q = q'$

Thus, the two sets intersect meaning they must be the same $\implies$ the sets $f(\Omega) + q$ must be disjoint.

Now, assume that $f(\Omega)$ is measurable. We also have that $\bigcup_{q \in \mathbb{Q} \cap [-1,1]} f(\Omega) + q$ must be measurable (as the countable union of translated, measurable sets). By step 3 and additivity and translation invariance:

$$m(\bigcup_{q\in\mathbb{Q}\cap[-1,1]} f(\Omega)+q) = \sum_{q\in\mathbb{Q}\cap[-1,1]} m(f(\Omega))$$

As the countable sum of some number, it must either be 0 or ∞. However, as a subset of $[k_0-1, k_0+2]$, the measure of the set must be smaller than 3 and thus it can only be 0.

However, we showed previously that A_{k_0} is a subset of this union with non-zero measure. Thus, the union's measure must be >0 ↯. Thus, the set must be non-measurable. □

1.3 The Cantor Set

Define the **Cantor Set** as the set $C=\bigcap_{k=1}^{\infty} C_k$ where $C_1=[0,\frac{1}{3}]\cup[\frac{2}{3},1]$ and $\forall k\geq 2:$

$$C_k=\bigcup_{j=1}^{2^k} I_{k,J}$$

where $\forall j\in\{1...,2^{k-1}\}$, $I_{k,2j-1}$ and $I_{k,2j}$ are the first and final third of the interval $I_{k-1,j}$.

Theorem 1.3. The Cantor set is closed, uncountable and has outer measure 0.

Proof. It being closed follows directly from it being the intersection of closed sets.

Then, $\forall k$ we have that $m^*(C)\leq m^*(C_k)$ since $C\subseteq C_k$. Then:

$$m^*(C_k)=\sum_{k=1}^{2^k} l(I_{k,j})=\sum_{k=1}^{2^k} 3^{-k}=(\frac{2}{3})^k$$

Which goes to 0 as $k\to\infty$. Thus, $m^*(C)$ is 0 and also measurable.

Finally, to show it is uncountable, assume it is countable and thus has some enumeration $(x_n)_{n\in\mathbb{N}}$. Then, define the following sequence of intervals:

$$I_1=\begin{cases}[0,\frac{1}{3}], & \text{if } x_1\in[\frac{2}{3},1],\\ [\frac{2}{3},1], & \text{otherwise.}\end{cases}$$

Then, $\forall k\geq 2$, we can take I_k to be a sub-interval (one of those from C_k) of I_{k-1} that x_k is not in. By the Nested Interval Property, the intersection of all these intervals cannot be empty. However, the intersection is a subset of C and thus any point in the intersection must be some x_{n*}. Yet, we purposely constructed this sequence of intervals to avoid every element of C, including x_{n*} ↯. □

A modified version of the Cantor set involves removing intervals of length $\alpha 3^{-k}, 0<\alpha<1$. This set can have positive measure.

1.4 Can all measurable sets be obtained as combinations of complements and countable unions of open sets?

Def. A σ**-algebra** is a collection of sets ϕ such that:

1. $\mathbb{R} \in \phi$
2. $\forall C_1, C_2 \in \phi : C_1 \setminus C_2 \in \phi$
3. $\forall (C_k)_{k\in\mathbb{N}} \in \phi, \bigcup_{k=1}^{\infty} C_k \in \phi$

Proposition 1.4. Any intersection of σ-algebras is a σ-algebra as well.

Def. A **Borel set** is a set that is in the intersection of all σ-algebras that contain the open sets.

Remark. Proposition: There exists a subset of the Cantor set that is not Borel.

Def. The **Cantor-Lebesgue function** (Cantor staircase function) is a function $\phi : [0,1] \to [0,1]$

$$\phi(x) = \begin{cases} \frac{1}{2}, & \text{if } x \in (\frac{1}{3}, \frac{2}{3}), \\ \frac{1}{4}, & \text{if } x \in (\frac{1}{9}, \frac{2}{9}), \\ \frac{3}{4}, & \text{if } x \in (\frac{7}{9}, \frac{8}{9}), \\ \dots, & \\ \frac{i}{2^k}, & \text{if } x \in J_{k,i}, \end{cases}$$

Where $J_{k,i}$ is the i-th interval of $[0,1] - C_k$. We also have that $\phi(0) = 0$ and $\forall y \in C - \{0\}$:

$$\phi(y) = \sup\{\phi(x) : x \in [0, y) - C\}$$

This property maintains monotonicity and the set in question is not empty since $(\frac{1}{3^k}, \frac{2}{3^k}) \subseteq [0, y) - C]$ for k large.

Remark. Proposition. ϕ is increasing and continuous.

Proof. It being increasing follows directly from the definition.

It is trivially continuous on $[0,1] - C$ since it is piecewise constant. Otherwise, let $x \in C$.

By construction, we have that $\forall k \in \mathbb{N} : \exists a_k, b_k \in [0,1] - C \cup \{0,1\}$ such that $a_k \leq x \leq b_k$ and $\phi(a_k) = \phi(b_k) - \frac{1}{2^k}$.

If $x = 0$, take $a_k = 0, b_k = J_{k,1}$, if $x = 1$ take $a_k = J_{k,2^k-1}, b_k = 1$.

$\phi(1) = 1$ since all values ≤ 1 and $\frac{2^k-1}{2^k} \to 1$ as $k \to \infty$.

Let $\epsilon > 0$. Then $\exists k_\epsilon$ such that $0 < e^{-k_\epsilon} < \epsilon$ and $\exists \delta_\epsilon > 0$ such that $(x - \delta_\epsilon, x + \delta_\epsilon) \cap [0,1] \subseteq [a_{k_\epsilon}, b_{k_\epsilon}]$. Then $\forall y \in (x - \delta_\epsilon, x + \delta_\epsilon)$, we have that $|\phi(y) - \phi(x)| < \phi(b_{k_\epsilon}) - \phi(a_{k_\epsilon}) = \frac{1}{2^{k_\epsilon}} < \epsilon$ □

Now define $\psi(X) = \phi(x) + x$. We then have that:

- ψ is strictly increasing since $\forall x < y : \psi(x) = x + \phi(x) < y + \phi(y) = \psi(y)$
- ψ is continuous since it is the sum of 2 continuous functions
- Bijective (injective from strict monotonicity and surjectivity from the Intermediate Value Theorem)

From this and A2, we have that ψ^{-1} is continuous and so ψ sends closed/open subsets of $[0,1]$ to closed/open subsets of $[0,2]$. In particular $\psi(C)$ is closed and thus measurable. Then

$$m(\psi(C)) = m([0,2] - ([0,2] - \psi(C)))$$

By excision, and since the second term is just $(\psi([0,1] - C))$, we get

$$= m([0,2]) - m(\psi([0,1] - C))$$

The second term is just

$$m(\psi((\frac{1}{3}, \frac{2}{3}))) + (\psi((\frac{1}{9}, \frac{2}{9}))) + \dots$$

$$= m(\frac{1}{2} + (\frac{1}{3}, \frac{2}{3})) + m(\frac{1}{4} + (\frac{1}{9}, \frac{2}{9}))$$

which is the same as just the measure of the interval by translation invariance. Thus, combining everything we get:

$$= 2 - \frac{1}{3} - \frac{2}{9} = 2 - \sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k} = 2 - \frac{1}{3} \sum_{k=1}^{\infty} \frac{2^k}{3^k} = 1$$

Thus, since $m(\psi(C)) > 0, \exists E \subseteq \psi(C)$ that is not measurable. Let $D = \psi^{-1}(E) \subseteq C \implies m^*(D) = 0$ so D is measurable.

If D were Borel, $\psi(D) = E$ would be Borel as well (from A2, we have that if ψ^{-1} is continuous, it maps Borel sets to Borel sets) and thus measurable, but it is not. Thus, D is not Borel.

2 Lebesgue Measurable Functions

Denote $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$ and $[0,\infty] = [0,\infty) \cup \{\infty\}$

Proposition 2.1. Let $A \subseteq \mathbb{R}$ be measurable and $f : A \to \overline{\mathbb{R}}$. The following are equivalent:

1. $\forall c \in \mathbb{R} : f^{-1}((c,\infty])$ is measurable.
2. $\forall c \in \mathbb{R} : f^{-1}([c,\infty])$ is measurable.
3. $\forall c \in \mathbb{R} : f^{-1}([-\infty, c))$ is measurable.
4. $\forall c \in \mathbb{R} : f^{-1}([-\infty, c])$ is measurable.

If all are satisfied, we say that f is measurable.

Proof. The transitive proof goes as follows:

1. $\implies$ 2. Follows from $f^{-1}([c,\infty]) = \bigcap_{n=1}^{\infty} f^{-1}((c-\frac{1}{n},\infty])$ and measurability is preserved by countable intersection.

2. $\implies$ 3. Follows from $f^{-1}([-\infty,c)) = A - f^{-1}([c,\infty])$.

3. $\implies$ 4. Follows from $f^{-1}([-\infty,c]) = \bigcap_{n=1}^{\infty} f^{-1}([-\infty,c+\frac{1}{n}))$.

4. $\implies$ 1. Follows from $f^{-1}((c,\infty]) = A - f^{-1}([-\infty,c])$ □

Remark. Let $A \subseteq \mathbb{R}$ measurable and $f : A \to \overline{\mathbb{R}}$

1. If f measurable $\implies \forall B \subseteq \mathbb{R}$ Borel, $f^{-1}(B)$ is measurable.
2. In case f is finite valued, ($f(A) \subseteq \mathbb{R}$) then f is measurable iff $\forall B \subseteq \mathbb{R}$ Borel, $f^{-1}(B)$ is measurable.

Proof. For the first statement: Define $\phi = \{B \subseteq \mathbb{R}, f^{-1}(B)$ is measurable$\}$. We show that it is a σ-algebra. To do so, it suffices to show that it contains the open sets.

Let O be open. Then $\exists((a_k,b_k))_{k\in\mathbb{N}}$ open intervals such that $O = \bigcup_{k=1}^{\infty}(a_k,b_k)$ and $(a_k,b_k) = [-\infty,b_k) \cap (a_k,\infty]$. It follows that $f^{-1}(O) = \bigcup_{k=1}^{\infty} f^{-1}([-\infty,b_k) \cap (a_k,\infty])$ so f^{-1} is measurable since both are measurable.

2. follows from the fact that $f^{-1}((c,\infty]) = f^{-1}((c,\infty))$ if $f(A) \subseteq \mathbb{R}$ and (c,∞) is Borel. □

Proposition 2.2. Let $A \subseteq \mathbb{R}$ measurable and $f : A \to \mathbb{R}$ continuous, then f is measurable.

Proof. If f is continuous, its inverse maps open sets to open sets.

In particular $f^{-1}((-\infty,c))$ is an open subset of $A \implies \exists O' \subseteq \mathbb{R}$ open such that $f^{-1}((-\infty,c)) = O' \cap A$. Since O' and A are measurable, so is $f^{-1}((-\infty,c))$. □

Remark. Measurable functions don't need to be continuous anywhere (ex: characteristc function of rational numbers).

Remark. There is a D and f measurable (even if f is continuous) such that $f^{-1}(D)$ is not measurable. Simply take D to be the non Borel subset of the Cantor set we constructed. Then, we have that $E = \psi(D) \implies f = \psi^{-1}$ is continuous and $f^{-1}(D) = \psi(D) = E$ is non-measurable.

Def. Let $A \subseteq \mathbb{R}$ be measurable and $P(x)$ be a statement depending on $x \in A$. We say that $P(x)$ is true **almost everywhere (AE)** in A or for **almost every** $x \in A$ if $m(\{x \in A : P(x) \text{ is false}\}) = 0$.

Remark. Let $(P_n(x))_{n\in\mathbb{N}}$ be a countable collection of statements depending on $x \in A$. Then:

$$[\forall n \in \mathbb{N}, P_n(x) \text{ is true a.e.}] \iff [\text{A.e.}, \forall n \in \mathbb{N}, P_n(x) \text{ is True}]$$

Proof.

$$m(\{x \in A : [\forall n \in \mathbb{N} : P_n(x)] \text{ is false}\})$$
$$= m(\{x \in A : \exists n \in \mathbb{N}, [P_n(x) \text{ is false}]\})$$

$$= m(\bigcup_{n\in\mathbb{N}} \{x \in A : [P_n(x) \text{ is false}]\})$$

$$\leq \sum_{n=1}^{\infty} m(\{x \in A : [P_n(x) \text{ is false}]\}) = 0$$

For the LHS to have measure 0, it must be the case that $\forall n \in \mathbb{N} : m(\{x \in A : [P_n(x) \text{ is false}]\}) = 0$ □

Proposition 2.3. Let $f : A \to \overline{\mathbb{R}}$ be measurable (implying that A) is measurable and let $g : A \to \overline{\mathbb{R}}$ be such that $f = g$ a.e. in A. Then g is measurable as well.

Proof. $\forall c \in \mathbb{R}, g^{-1}([-\infty, c]) = (g^{-1}([-\infty, c) \cap N) \cup g^{-1}([-\infty, c)) \cap (A - N)$, where $N = \{x \in A : f(x) \neq g(x)\}$.

The first term is measurable as a subset of a set (N) with measure 0 and the second is measurable since f is measurable. Thus, their union $g^{-1}([-\infty, c))$ is also measurable. □

Proposition 2.4. Let $(A_n)_{n\in\mathbb{N}}$ be disjoint, measurable sets with $A = \bigcup_{n\in\mathbb{N}} A_n$. Let $(f_n)_{n\in\mathbb{N}}, f_n : A_n \to \overline{\mathbb{R}}$ be measurable. Then, let $f = f_n$ on A_n is also measurable.

Proof. Follows from $f^{-1}([-\infty, c)) = \bigcup_{n\in\mathbb{N}} f_n^{-1}([-\infty, c))$ being a countable union of measurable sets. □

Example 2.1. Examples of measurable functions include:

1. Piecewise continuous functions
2. Characteristic functions and simple functions

Def. A **simple function** is a function $f : A \to \mathbb{R}$ that takes on finitely many values. We denote the **canonical representation** of f as:

$$f = \sum_{n=1}^{N} \alpha_n \chi_{A_n}(x)$$

where $A_n = f^{-1}(\alpha_n)$.

Proposition 2.5. Let $A \subseteq \mathbb{R}$ be measurable:

1. $\forall B \subseteq A$ measurable, $f|_B$ (i.e. the function defined only on B) is measurable.
2. $\forall B \subseteq \mathbb{R}$ Borel, $f : B \to \mathbb{R}$ continuous and $g : A \to B$ measurable gives that $f \circ g$ is measurable.
3. For $f : A \to \overline{\mathbb{R}}, g : A \to \mathbb{R}$ with both measurable, we have that $g + f$ is measurable.
    - We avoid $\overline{\mathbb{R}}$ for the second function to avoid $\infty - \infty$
4. For $f, g : A \to \mathbb{R}$ measurable, we have that $f \cdot g$ is measurable.

5. For finitely many measurable functions $f_n : A \to \overline{\mathbb{R}}$, we have that both their max and min are measurable.

Proof. The proof for each statement is as follows:

1. $\forall c \in \mathbb{R} : f|_B^{-1}([-\infty, c)) = B \cap f^{-1}([-\infty, c))$. Since both B and the inverse are measurable, so is their intersection.
2. $\forall c \in \mathbb{R} : f^{-1}((-\infty, c))$ is Borel since f is continuous on Borel sets and $(-\infty, c)$ is open. Thus, $g^{-1}(f^{-1}((-\infty, c))$ is measurable since g is measurable.
3. $\forall c \in \mathbb{R}, x \in A : (f+g)(x) < c \iff f(x) < c - g(x) \iff \exists q \in \mathbb{Q} : f(x) < q < c - g(x)$.

$$\implies (f+g)^{-1}([-\infty, c)) = \bigcup_{q \in \mathbb{Q}} f^{-1}([-\infty, q)) \cap g^{-1}([-\infty, c-q))$$

Since both elements of the intersection are measurable and we are taking their countable union, the whole set is measurable.

4. $f \cdot g = \frac{1}{2}[(f+g)^2 - f^2 - g^2]$. Then, since $f, g, f+g$ and taking the square is a continuous function, we have that $f \cdot g$ is continuous.
5. $\forall c \in \mathbb{R}$ we have that;

$$\max(f_1 ... f_n)^{-1}([-\infty, c)) = \bigcap_{k=1}^{n} f_k^{-1}([-\infty, c))$$

and

$$\min(f_1 ... f_n)^{-1}([-\infty, c)) = \bigcup_{k=1}^{n} f_k^{-1}([-\infty, c))$$

□

Remark. There are f, g measurable such that $f \circ g$ is not measurable. For example, take D to be the measurable, non-Borel subset of the Cantor set we constructed with $E = \psi(D)$. We have that χ_D and $\psi^{-1}()$ are both measurable.

However $(\chi_D \circ \Psi^{-1})^{-1}((\frac{1}{2}, \infty]) = \psi(\chi_D^{-1}((\frac{1}{2}, \infty])) = \psi(D) = E$.

Since E is not measurable, neither is the composition of the two functions.

Def. Let $(f_n)_{n \in \mathbb{N}}, f_n : A \to \overline{\mathbb{R}}$ and $f : A \to \overline{\mathbb{R}}$. We say that:

1. $(f_n)_{n \in \mathbb{N}}$ **converges pointwise** to f in $B \subseteq A$ if $\forall x \in B : \lim_{n \to \infty} f_n(x) = f(x)$
2. $(f_n)_{n \in \mathbb{N}}$ **converges uniformly** to f in $B \subseteq A$ if $|f| < \infty$ and $\lim_{n \to \infty} \sup_B |f - f_n| = 0$.

Proposition 2.6. Let $(f_n)_{n \in \mathbb{N}}, f_n : A \to \overline{\mathbb{R}}$ be measurable functions that converge pointwise a.e. in A to f. Then, f is measurable.

Proof. Define $N = \{x \in A : \lim_{n\to\infty} f_n(x) \neq f(x)\}$. Then, $\forall c \in \mathbb{R}$:

$$f^{-1}([-\infty, c)) = f^{-1}(([-\infty, c)) \cap N) \cup f^{-1}(([-\infty, c)) \cap (A - N))$$

The former set has measure 0 as a subset of N. As for the latter, $\forall x \in A - N, f(x) \leq c$:

$$\forall k \in \mathbb{N} : \exists n_k \in \mathbb{N} : \forall n \geq n_\epsilon : f_n(x) \leq c + \frac{1}{k}$$

$$\implies f^{-1}(([-\infty, c)) \cap (A - N)) = \bigcap_{k=1}^{\infty} \bigcup_{n_k=1}^{\infty} \bigcap_{n=n_k}^{\infty} f_n^{-1}([-\infty, c + \frac{1}{k}])$$

Hence, as the countable intersection of the countable union of the countable intersection of measurable sets, $f^{-1}(([-\infty, c)) \cap (A - N))$ is measurable as well. Thus, we have that $f^{-1}([-\infty, c))$ is measurable. □

Lemma 2.7. Simple Approximation Lemma: Let $f : A \to \mathbb{R}$ be measurable and bounded ($|f| < M$). Then, if $\forall \epsilon > 0, \exists \Phi_\epsilon, \Psi_\epsilon : A \to \mathbb{R}$ simple functions with $\Phi_\epsilon \leq f \leq \Psi_\epsilon \leq \Phi_\epsilon + \epsilon$, we have that both $\Phi_\epsilon, \Psi_\epsilon \to f$ uniformly in A.

Proof. Let $n_\epsilon \in \mathbb{N}$ be such that $\frac{2M}{n_\epsilon} < \epsilon$. Then, partition the range by letting $y_k = M(\frac{2k}{n_\epsilon} - 1)$ so that $-M = y_0 < .. < y_{n_\epsilon} = M$ and $y_k = y_{k-1} + \frac{2M}{n_\epsilon}$.

Let $A_k = f^{-1}([y_{k-1}, y_k))$ so A is the union of these disjoint sets and $A_k = f^{-1}((-\infty, y_k)) \cap f^{-1}([y_{k-1}, \infty))$, meaning that the A_k are measurable.

Define $\Phi_\epsilon = \sum_{k=1}^{n_\epsilon} y_{k-1}\chi_{A_k}$ and $\Psi_\epsilon = \sum_{k=1}^{n_\epsilon} y_k \chi_{A_k} = \Phi_\epsilon + \frac{2M}{n_\epsilon} < \epsilon$.

We then have that $\Phi_\epsilon \leq f \leq \Psi_\epsilon \leq \Phi_\epsilon + \epsilon$ in A since $y_{k-1} \leq f \leq y_k$ in A_k for all k. □

Theorem 2.8. Simple Approximation Theorem: Let $f : A \to \overline{\mathbb{R}}$ with A measurable. Then, f is measurable iff $\exists$ simple functions $(\Phi_n)_{n\in\mathbb{N}}$ such that:

1. $(\Phi_n)_{n\in\mathbb{N}}$ converges pointwise to f in A
2. $\Phi_n \leq |f|$ in A for all $n \in \mathbb{N}$

Moreover, if $f \geq 0$ in A, we can choose $\Phi_n \geq 0$ with $\Phi_{n+1} \geq \Phi_n, \forall n \in \mathbb{N}$

Proof. We begin by showing that it is the case for f positive, then we extend to general functions.

Case $f \geq 0$: Let $f_n = \min(f, n), \forall n \in \mathbb{N}$. Then f_n is bounded and measurable so the SAL gives us that $\exists \Phi_n$ simple such that $\Phi_n \leq f_n \leq \Phi_n + \frac{1}{n}$ in A.

Define $\Phi_n^* = \max(\Phi_1 ... \Phi_n, 0)$ so that $\Phi_n^* \geq 0$ and is increasing. Moreover, $\Phi_n^* \leq f_n \leq f$ so 2. holds.

Finally $\forall x \in A$:

- If $f(x) \leq \infty$ then $\forall n \geq f(x), f_n(x) = f(x)$ and so $0 \leq f(x) - \Phi_n^*(x) \leq \frac{1}{n} \implies \lim_{n\to\infty} \Phi_n^*(x) = f(x)$
- If $f(x) = \infty$, then $\forall n \in \mathbb{N}, f_n(x) = n$ so $\Phi_n^* > n - \frac{1}{n} \to \infty \implies \lim_{n\to\infty} \Phi_n^*(x) = f(x)$

Thus, we have pointwise converge of Φ_n^* to f.

Case f general: Write $f = f_+ - f_-$ where $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$.

Then $\exists (\Phi_{n_{+,-}})_{n\in\mathbb{N}}$ simple functions such that they converge to $f_{+,-}$ and $0 \leq \Phi_{n_{+,-}} \leq f_{+,-}$. We have that $\Phi = \Phi_+ - \Phi_-$ is a simple function as well.

Moreover $(\Phi_{n,+} - \Phi_{n,-})_{n\in\mathbb{N}}$ converge pointwise to $f_+ - f_- = f$ and $|\Phi_n| \leq \Phi_{n,+} - \Phi_{n,-} \leq f_+ - f_- = f$.

The other direction follows directly from simple functions being measurable and the pointwise limit of measurable functions being measurable as well.

□

A function is **Lebesgue** measurable if $\forall B \subseteq \mathbb{R}$ Borel, $f^{-1}(B)$ is **Lebesgue** measurable.

A function is **Borel** measurable if $\forall B \subseteq \mathbb{R}$ Borel, $f^{-1}(B)$ is Borel.

Theorem 2.9. Egoroff's Theorem: Let $A \subseteq \mathbb{R}$ be measurable and $m(A) < \infty$. Let $(f_n)_{n\in\mathbb{N}}$ be measurable functions that converge pointwise to f. Then, $\forall \epsilon > 0 : \exists F_\epsilon \subseteq A$ closed such that:

1. $(f_n)_{n\in\mathbb{N}}$ converge uniformly on F_ϵ
2. $m(A - F_\epsilon) < \epsilon_n$

Proof. The proof can be broken down into 2 steps:

First, let $\epsilon_n, \delta_n > 0$ for $n \in \mathbb{N}$. Then $\exists A_n \subseteq A$ measurable and $k_n \in \mathbb{N}$ such that:

1. $|f_k - f| \leq \delta_n$ in $A_n, \forall k \geq k_n$
2. $m(A - A_n) < \epsilon$

Then, define:

$$A_{n,k} = \{x \in A : |f_j - f| < \delta_n, \forall j \geq k\} = \bigcap_{j=k}^{\infty} (|f_j - f|)^{-1}((-\infty, \delta_n))$$

We have that $A_{n,k}$ is measurable since f_j, f are measurable and $|\cdot|$ is continuous. Moreover, we have that $\bigcup_{k\in\mathbb{N}} A_{n,k} = A$ since $f_n \to f$ pointwise in A.

With $A_{n,k} \subseteq A_{n,k+1}, \forall k$, we get $\lim_{k\to\infty} m(A_{n,k}) = m(A)$.

In particular, $\exists k_n \in \mathbb{N}$ with $m(A - A_{n,k_n}) = m(A) - m(A_{n,k_n}) < \epsilon_n$ (for this step, we need $m(A) < \infty$).

Thus, for $A_n = A_{k,k_n}$ we have:

$$\begin{cases} |f_k - f| < \delta_n, & \forall k \geq k_n, \\ m(A - A_n) < \epsilon_n \end{cases}$$

Finally let $F = \bigcap_{n\in\mathbb{N}} A_n$. Then F is measurable as the countable intersection of measurable sets. Choose $\delta_n = \frac{1}{n}$ and $\epsilon_n = \frac{\epsilon}{2^{n+1}}$

Since F is measurable, $\exists F' \subseteq F$ closed such that $m(F - F') = \frac{\epsilon}{2}$. Then, in F', we have that $|f_n - f| \leq \frac{1}{n}, \forall n \in \mathbb{N}$ since $F \subseteq A_n$. Thus, the f_n converge uniformly in F' to f.

$$m(A - F') = m(A - F) + m(F - F') = m(\bigcup_{n\in\mathbb{N}} (A - A_n)) + \frac{\epsilon}{2} \leq \epsilon$$

□

Theorem 2.10. Lusin's Theorem: Let $f : A \to \mathbb{R}$ be measurable. Then, $\forall \epsilon > 0 : \exists F_\epsilon \subseteq A$ closed such that:

1. f is continuous on F_ϵ
2. $m(A - F_\epsilon) < \epsilon$

Proof. We begin by considering the case where f is simple and extend the result to general cases

Case f simple: Since the A_ks of the simple functions are measurable, each contains some F_k closed with $m(A_K - F_k) < \frac{\epsilon}{N}$. Then, $F = \bigcup_{k=1}^N F_k$ is closed as well as a finite union of closed sets and f is continuous on F since it is constant.

Finally, $m(A - F) \leq \sum_{k=1}^N m(A - F_k) < \epsilon$.

We have proven the case for f simple

For the case of f measurable and $m(A) < \infty$. By the Simple Approximation Theorem, $\exists (\psi_n)_{n\in\mathbb{N}}$ simple functions such that $\psi_n \to f$ pointwise in A. For each ψ_n, since they are simple, there is a closed set $F_n \subseteq A$ such that ψ_n is continuous on F_n and $m(A \setminus F_n) < \epsilon 2^{-n-1}$ (by the first case).

If we take $F = \bigcap_{n\in\mathbb{N}} F_n$ it is closed as well and we have a sequence of continuous functions on this set. If they converged uniformly, we would have that their limit is continuous as well.

Using Egoroff's Theorem, $\exists F_0 \subseteq A$ closed such that $\psi_n \to f$ uniformly on F_0 and $m(A \setminus F_0) < \frac{\epsilon}{2}$. Then, f is continuous on F as the uniform limit of continuous functions and $m(A \setminus F) \leq \sum_{n=0}^\infty m(A \setminus F_n) < \epsilon$

□

The case of sets with infinite measure holds for Lusin's Theorem.

3 The Lebesgue Integral

3.1 Simple functions on a set of finite measure

Def. Let $\psi : A \to \mathbb{R}$ be a simple function on A (where the measure of A is finite) and let $\psi = \sum_{k=1}^{N} a_k \chi_{A_k}$ be its canonical representation. We define the integral of ψ over A, denoted by $\int_A \psi$ or $\int_A \psi(x)dx$ as the number $\int_A \psi = \sum_{k=1}^{N} a_k m(A_k)$.

For every $B \subseteq A$ measurable, we denote $\int_B \psi = \int_B \psi|_B$.

Remark. Let $\psi_{\alpha,\beta}(x) = \alpha$ if $x \in \mathbb{Q} \cap [a,b]$ and $\psi_{\alpha,\beta}(x) = \beta$ if $x \notin \mathbb{Q} \cap [a,b]$ for $\alpha, \beta \in \mathbb{R}, a < b \in \mathbb{R}$

Then $\psi_{\alpha,\beta} = \alpha \chi_{\mathbb{Q} \cap [a,b]} + \beta \chi \mathbb{R} \setminus \mathbb{Q} \cap [a,b]$ and so we have that $\int_{[a,b]} \psi_{\alpha,\beta} = \alpha m(\mathbb{Q} \cap [a,b]) + \beta m(\mathbb{R} - \mathbb{Q} \cap [a,b]) = \beta(b-a)$.

3.2 Measurable and bounded functions on a set of finite measure

Def. Let A be a measurable set with finite measure and $f : A \to \mathbb{R}$ be a function. We say that f is **(Lebesgue) integrable** over A if $\underline{\int_A} f$ (lower) $= \overline{\int_A} f$ (upper). Where

$$\underline{\int_A} f = \sup \left\{ \int_A \phi : \phi \text{ simple }, \phi \leq f \text{ on } A \right\}$$

$$\overline{\int_A} f = \inf \left\{ \int_A \phi : \phi \text{ simple }, \phi \geq f \text{ on } A \right\}$$

These sets are non-empty since f is bounded meaning there must be some $c > 0$ such that $-c\chi_A \leq f \leq c\chi_A$

We then denote $\int_A f = \int_A f(x)dx = \underline{\int_A} f = \overline{\int_A} f$ and we call $\int_A f$ the **integral of** f **over** A. Once again, for every $B \subseteq A$ measurable, we denote $\int_B f = \int_B f|_B$.

Remark.

$$-\infty < \int_A f < \infty \text{ since } -cm(A) = \int_A -c\chi_A \leq \int_A f \leq \int_A c\chi_A$$

Proposition 3.1. If $f : [a,b] \to \mathbb{R}$ is Riemann integrable, then f is Lebesgue integrable.

$$A \subseteq B$$

Proof. Remark that $\{\sum_{i=1}^{N} \inf_{(x_{i-1},x_i)} f * (x_i, x_{i-1}) : a = x_0 < ... < x_N = b\} \subset \{\int_{[a,b]} \phi : \phi \text{ simple }, \phi \leq f\}$ and similarly for the other set with the supremum.

The functions $\int \sum_{i=1}^{N} \inf_{(x_{i-1},x_i)} f \chi_{[x_{i-1},x_i)}$ are the functions we are integrating (step-functions which are simple).

Use that the supremum of the smaller set will be smaller than the supremum of the larger set.

Hence,

$$\underline{\int_{[a,b]}^{R}} f \leq \underline{\int_{[a,b]}^{L}} f \leq \overline{\int_{[a,b]}^{L}} f \leq \overline{\int_{[a,b]}^{R}} f$$

So if the 2 on the outside are equal $\underline{\int_{[a,b]}^{R}} f = \overline{\int_{[a,b]}^{R}} f$ and thus Riemann integrable, then the inner ones are equal as well $\underline{\int_{[a,b]}^{L}} f = \overline{\int_{[a,b]}^{L}} f$.

□

Theorem 3.2. Let $f : A \to \mathbb{R}$ be measurable and bounded on A with measure of A finite. Then f is **integrable**.

Proof. From the Simple Approximation Lemma. $\exists (\phi_n)_{n\in\mathbb{N}}$ of simple functions such that $\phi_n \leq f \leq \phi_n + \frac{1}{n}, \forall n \in \mathbb{N}$. It follows that:

$$\int_A \phi_n \leq \underline{\int_A} f \leq \overline{\int_A} f \leq \int_A \phi_n + \frac{1}{n}$$

Let $\phi_n = \sum_{k=1}^{N_n} a_{k,n} \chi_{A_{k,n}}$ be the canonical representation of ϕ_n. Then $\phi_n + \frac{1}{n} = \sum_{k=1}^{N_n} (a_{k,n} + \frac{1}{n}) \chi_{A_{k,n}}$ is the canonical representation of $\phi_n + \frac{1}{n}$ and so

$$\int_A \phi + \frac{1}{n} = \sum_{k=1}^{N_n} (a_{k,n} + \frac{1}{n}) m(A_{k,n}) = \sum_{k=1}^{N_n} a_{k,n} m(A_{k,n}) + \frac{1}{n} m(A_{k,n}) = \int_A \phi_n + \frac{m(A)}{n}$$

It follows that:

$$0 \leq \overline{\int_A} f - \underline{\int_A} f \leq \frac{m(A)}{n} \to_{n\to\infty} 0$$

so we have that $\overline{\int_A} f = \underline{\int_A} f$

□

Proposition 3.3. Let $f, g : A \to \mathbb{R}$ measurable and bounded, $m(A) < \infty$:

1. $\forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g$ is measurable and bounded with $\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$
2. If $f \leq g$ on A then $\int_A f \leq \int_A g$
3. $|f|$ is measurable and bounded and $|\int_A f| \leq \int_A |f|$
4. $\forall B \subseteq A$ measurable, $f \chi_B$ is measurable and bounded and $\int_A f \chi_B = \int_B f$
5. $\forall A_1, A_2 \subseteq A$ measurable and disjoint $\int_{A_1 \cup A_2} = \int_{A_1} f + \int_{A_2} f$. Moreover, if $m(A_2) = 0$, then $\int_{A_2} f = 0$ so $\int_{A_1 \cup A_2} = \int_{A_1} f$

Lemma 3.4. Independence on the representation: Let $n \in \mathbb{N}, a_1 ... a_n \in \mathbb{R}$ and $A_1 ... A_n \subseteq A$ be measurable and disjoint. Then

$$\int_A \sum_{k=1}^{n} a_k \chi_{A_k} = \sum_{k=1}^{n} a_k m(A_k)$$

Proof. If this is the canonical representation, we have nothing to prove. Otherwise, if the union of the A_k is not A, take $a_0 = 0$ and $A - \cup_{k=1}^{n} A_k$ so that $\sum_{k=1}^{n} \chi_{A_k} = \sum_{k=0}^{n} \chi_{A_k}$ and $\cup_{k=1}^{n} A_k = A$

Let $\Psi = \sum_{k=0}^{n} \chi_{A_k}$. Let $n' \in \mathbb{N}$ and $a'_1 < ... < a'_{n'}$ be such that they contain all the a_k (i.e. they are equal to $\Psi(\mathbb{R})$).

Let $J_k = \{j \in \{0....n\} : a_j = a'_k\}$ and $A_{k'} = \bigcup_{j \in J_k} A_j = \Psi^{-1}(a'_k)$ so that $\bigcup_{k=1}^{n'} A_{k'} = \bigcup_{k=1}^{n'} = \bigcup_{j \in J_k} A_j = A$

Moreover $\forall k_1 \neq k_2, A'_{k_1} \cap A'_{k_2} = \bigcup_{j_1 \in A_{k_1}} \bigcup_{j_2 \in A_{k_2}} A_{j_1} \cap A_{j_2} = \emptyset$ since $J_{k_1} \cap J_{k_2} = \emptyset$.

This gives that $\sum_{k=1}^{n} a_{k'} \chi_{A_{k'}}$ is the canonincal representation of Ψ. It follows that:

$$\int_A \Psi = \sum_{k=1}^{n'} a_{k'} m(A_{k'}) = \sum_{k=1}^{n'} a_{k'} \sum_{j \in J_k} m(A_j) = \sum_{k=1}^{n'} \sum_{j \in J_k} a_j m(A_j) = \sum_{j=0}^{n} a_j m(A_j) = \sum_{j=1}^{n} a_j m(A_j)$$

□

Proof. We first consider the case where f, g are simple functions.

Let $f = \sum_{k=1}^{n} a_k \chi_{A_k}$ and $g = \sum_{k=1}^{n'} a'_k \chi_{A'_k}$ be their canonical representations. Let $A_{i,j} = A_i \cap A'_j$ and $a_{i,j} = a_i$, $a'_{i,j} = a'_j$ for all i, j so that:

$$f = \sum_{i=1}^{n} \sum_{j=1}^{n'} a_{i,j} \chi_{A_{i,j}}$$

$$g = \sum_{i=1}^{n} \sum_{j=1}^{n'} a'_{i,j} \chi_{A_{i,j}}$$

Moreover, the sets $A_{i,j}$ are disjoint since the sets A_i are disjoint and the sets A'_j are disjoint.

1. $\forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g = \sum_{i=1}^{n} \sum_{j=1}^{n'} (\alpha a_{i,j} + \beta a'_{i,j}) \chi_{A_{i,j}}$ is a simple function and by applying the Lemma

$$\int_A \alpha f + \beta g = \sum_{i=1}^{n} \sum_{j=1}^{n'} (\alpha a_{i,j} + \beta a'_{i,j}) m(A_{i,j})$$

$$= \alpha \sum_{i=1}^{n} \sum_{j=1}^{n'} a_{i,j} m(A_{i,j}) + \beta \sum_{i=1}^{n} \sum_{j=1}^{n'} a'_{i,j} m(A_{i,j}) = \alpha \int_A f + \beta \int_A g$$

2. $f \le g$ in $A \implies \forall i,j : a_{i,j} \le a'_{i,j}$ in $A_{i,j} \implies$

$$\sum_{i=1}^{n}\sum_{j=1}^{n'} a_{i,j}\chi_{A_{i,j}} \le \sum_{i=1}^{n}\sum_{j=1}^{n'} a'_{i,j}\chi_{A_{i,j}}$$

$$\int_A f \le \int_A g$$

3. $|f| = \sum_{j=1}^{n} |a_j|\chi_{A_j}$ is a simple function and from 2. we have that

$$-|f| \le f \le |f| \implies -\int_A |f| \le \int_A f \le \int_A |f| \implies |\int_A f| \le \int_A |f|$$

4. $\forall B \subseteq A$ measurable, $f\chi_B = \sum_{i=1}^{n}\sum_{j=1}^{n} a_j\chi_{A_j}\chi_B = \sum_{j=1}^{n} a_j\chi_{A_j\cap B}$ is a simple function.

   Moreover $f|_B = \sum_{j=1}^{n} a_j\chi_{A_j}|_B = \sum_{j=1}^{n} a_j\chi_{A_j\cap B}$.

   Thus, $\int_A f\chi_B = \int_B f$.

5. $\forall B_1, B_2 \subseteq A$ measurable and disjoint, $\int_{B_1\cup B_2} f = \int_A f\chi_{B_1\cup B_2}$ by 4.

$$= \int_A \sum_{j=1}^{n} a_j\chi_{A_j\cap(B_1\cup B_2)} = \sum_{j=1}^{n} a_j m(A_j\cap(B_1\cup B_2))$$

$$= \sum_{j=1}^{n} a_j m(A_j\cap B_1) + a_j m(A_j\cap B_2)$$

   by additivity since $B_1 \cap B_2 = \emptyset$

$$= \int \sum a_j\chi_{A_j\cap B_1} + \int \sum a_j\chi_{A_j\cap B_2}$$
$$= \int_A f\chi_{B_1} + \int_A f\chi_{B_2} = \int_{B_1} f + \int_{B_2} f$$

   If $m(B_2) = 0$ then $m(A_j\cap B_2) = 0$ and so $\int_{B_2} f = 0$.

Now we prove the proposition when f, g are bounded measurable but not necessarily simple.

1. $\forall \alpha, \beta \in \mathbb{R}, \alpha f + \beta g$ is bounded measurable as well.

$$\forall \alpha \in \mathbb{R}, \int_A \alpha f = \underline{\int_A} \alpha f = \sup\{\int_A \Phi : \Phi \text{ simple}, \Phi \le \alpha f\}$$

   There are multiple cases:

   - The integral is 0 if $\alpha = 0$.

- Now suppose $\alpha > 0$. Let $\Phi = \alpha\tilde{\Phi}$ and consider the set

$$\sup\left\{\int_A \alpha\tilde{\Phi} : \tilde{\Phi} \text{ simple, } \tilde{\Phi} \le f\right\} = \alpha \sup\left\{\int_A \tilde{\Phi} : \tilde{\Phi} \text{ simple, } \tilde{\Phi} \le f\right\} = \alpha \underline{\int_A} f$$

- For $\alpha < 0$, consider the set:

$$\inf\left\{\int_A \alpha\tilde{\Phi} : \tilde{\Phi} \text{ simple, } \tilde{\Phi} \ge f\right\} = \alpha \inf\left\{\int_A \tilde{\Phi} : \tilde{\Phi} \text{ simple, } \tilde{\Phi} \ge f\right\} = \alpha \overline{\int_A} f$$

In each case:

$$= \alpha \int_A f$$

We now show the property for summation:

$$\int_A (f+g) = \underline{\int_A} (f+g) = \sup\left\{\int_A \Phi : \Phi \text{ simple}, \Phi \le f+g \text{ on } A\right\} =$$

$$= \sup\left\{\int_A \Phi_1 : \Phi_1 \text{ simple}, \Phi_1 \le f \text{ on } A\right\} + \sup\left\{\int_A \Phi_2 : \Phi_2 \text{ simple}, \Phi_2 \le g \text{ on } A\right\}$$

$$\ge \sup\left\{\int_A \Phi_1 + \Phi_2 : \text{ both simple and } \Phi_1 + \Phi_2 \le f+g \text{ on } A\right\}$$

$$= \underline{\int_A} f + \underline{\int_A} g = \int_A f + \int_A g$$

Similarly, the upper integral of $f+g \le$ upper integral of f + the upper integral of g. Combining the two gives us equality.

Combining both arguments gives us the result for both.

2. To show $f \le g$ on A, we have that $\forall \phi$ simple: $\phi \le f \implies \phi \le g$.

$$\implies \int_A f = \underline{\int_A} f \le \underline{\int_A} g = \int_A g$$

3. $|f|$ bounded and measurable as well since f is bounded measurable and $x \to |x|$ is continuous. Then, using 2.

$$-|f| \le f \le |f| \implies -\int_A |f| \le \int_A f \le \int_A |f| \implies |\int_A f| \le \int_A |f|$$

4. $\chi_B f$ is bounded and measurable as the product of 2 bounded and measurable functions.

$$\int_A \chi_B f = \underline{\int_a} \chi_B f = \sup S_1$$

where

$$S_1 = \left\{ \int_A \phi : \phi \text{ simple and } \phi \leq f\chi_B \text{ on A} \right\}$$

For ϕ to belong to S_1, we need:

$$\phi = \begin{cases} \phi \leq f \text{ on } B \\ \phi \leq 0 \text{ on } A - B. \end{cases}$$

Take

$$S_2 = \left\{ \int_B \phi, \phi \text{ simple and } \phi \leq f \text{ on } B, \phi = 0 \text{ on } A - B \right\}$$

Then:

$$\sup S_1 \geq \sup S_2, \text{ from } (S_2 \subseteq S_1)$$

Then, we can decompose the integral in 2

$$\int_A \phi = \int_B \phi + \int_{A-B} \phi = \int_B \phi$$

Let $\tilde{\phi} = \phi|_B$, then

$$S_3 = \left\{ \int_B \tilde{\phi} \text{ simple and } \tilde{\phi} \leq f \text{ on } B \right\}$$

Gives us that $S_2 = S_3$ and $\sup S_3 = \underline{\int_A} f$. Putting it all together gives that.

$$\underline{\int_A} \chi_B f \geq \underline{\int_A} f$$

A similar process yields, $\overline{\int_A} \chi_B f \leq \overline{\int_A} f$ so we have that $\int_A \chi_B f = \int_B f$

5. $\forall A_1, A_2 \subseteq A$ measurable and disjoint

$$\int_{A_1 \cup A_2} f = \int_A f\chi_{A_1 \cup A_2} = \int_A f(\chi_{A_1} + \chi_{A_2}) = \int_A f\chi_{A_1} + \int_A f\chi_{A_2} = \int_{A_1} f + \int_{A_2} f$$

In the case $m(A_2) = 0$, since we have proven $\int_{A_2} \phi = 0$ for all simple functions we have that $\underline{\int_{A_2}} f = \overline{\int_{A_2}} f \implies \int_{A_2} f = 0$ (as the supremum/infimum of set with all 0s will be 0).

□

Theorem 3.5. Bounded convergence theorem: Let $A \subseteq \mathbb{R}$ measurable with finite measure. Let $(f_n)_{n\in\mathbb{N}}, f_n : A \to \mathbb{R}$ be measurable on A such that:

1. $\exists M > 0$ such that $\forall n \in \mathbb{N} : |f_n| \leq M$ on A
2. $\exists f : A \to \mathbb{R}$ such that $\forall x \in A : \lim_{n\to\infty} f_n(x) = f(x)$ (i.e. converges pointwise).

Then f is bounded measurable and

$$\lim_{n\to\infty} \int_A f_n = \int_A f$$

Proof. f is measurable since it is the pointwise limit of measurable functions f_n.

Moreover, it follows from 1. and 2. that $|f| \leq M$.

By Egoroff's theorem, $\forall \epsilon > 0 : \exists F_\epsilon \subseteq A$ measurable such that $f_n \to f$ converges uniformly on F_ϵ and $m(A - F_\epsilon) < \epsilon$

$$|\int_A f_n - \int_A f| = |\int_A f_n - f| \leq \int_A |f_n - f| = \int_{F_\epsilon} |f_n - f| + \int_{A-F_\epsilon} |f_n - f|$$

$$\leq \sup_{F_\epsilon} |f_n - f| * \int_{F_\epsilon} 1 + \sup_{A-F_\epsilon} |f_n - f| * \int_{A-F_\epsilon} 1$$

The first term goes to 0 as $n \to \infty$.

The second term is $m(F_\epsilon) < m(A)$

The third term is $\leq |f_n| + |f| \leq 2M$

The last term is $m(A - F_\epsilon) < \epsilon$

Thus, together:

$$\lim_{n\to\infty} \sup |\int_A f_n - \int_A f| \leq 2M\epsilon$$

As $\epsilon \to 0$, we obtain $\lim_{n\to\infty} |\int_A f_n - \int_A f| = 0$, i.e. $\lim_{n\to\infty} \int_A f_n = \int_A f$ □

Example 3.1. Take $f_n(x) = (cos(x))^n, \forall x \in (0,\pi)$. Then, $\forall x \in (0,1) : |f_n(x)| \leq 1, \lim_{n\to\infty} f_n(x) = 0$. Thus:

$$\lim_{n\to\infty} \int_{(0,\pi)} f_n = 0$$

Example 3.2. Take $f_n(x) = n\chi_{(0,\frac{1}{n})}$. Here, the sequence of functions does not satisfy uniform boundedness but does satisfy pointwise convergence. In fact, $\lim_{n\to\infty} f_n(x) = 0, \forall x \in (0,1)$.

However, $\forall n \in \mathbb{N} : \int_{(0,1)} f_n = 1 \neq 0$, thus the limit cannot be 0.

3.3 Case of nonnegative measurable functions

Includes functions that are unbounded and not necessarily on sets of finite measure. To deal with these functions, we bound the function and limit them to a set of finite measure. Then, we take the supremum over all these functions to get the integral.

Def. Let $A \subseteq \mathbb{R}$ be measurable (possibly of infinite measure) and $f : A \to \overline{\mathbb{R}^+}$ be measurable. We call **integral of f over A** and denote $\int_A f = \int_A f(x)dx$ the number:

$$\int_A f = \sup\left\{\int_B h : B \subseteq A, m(B) < \infty, h : B \to \mathbb{R} \text{ measurable, bounded and } 0 \le h \le f \text{ on } B\right\}$$

For every $B \subseteq A$, we denote $\int_B f = \int_B f|_B$

If $\int_A f < \infty$, we say that f is **integrable over A**.

Example 3.3. $\forall B \subseteq A$ measurable, $\int_A \chi_B = m(B)$

- $\forall B' \subseteq A, m(B') < \infty, h : B' \to \mathbb{R}$ measurable and bounded and $0 \le h \le \chi_B$ on B', then $\int_{B'} h \le \int_{B'} \chi_B = \int_A \chi_{B\cap B'} = m(B \cap B') \le m(B)$
- If B has finite measure, we trivially have the result. Otherwise, $\forall n \in \mathbb{N} : m(B \cap [-n, n]) \le m([-n, n]) < \infty$ and $\chi_{B\cap[-n,n]}$ is bounded measurable:

$$0 \le \chi_{B\cap[-n,n]} \le \chi_B$$

and

$$\int_{[-n,n]} \chi_{B\cap[-n,n]} = m(B \cap [-n, n]) \to m(B) \text{ as } n \to \infty$$

By continuity of measure. Thus, $\int_A \chi_B = m(B)$.

Proposition 3.6. Let $f, g : A \to \overline{\mathbb{R}^+}$ measurable:

1. $\forall \alpha, \beta \ge 0 : \alpha f + \beta g$ is nonnegative and measurable and $\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$
2. $f \le g$ on $A \implies \int_A f \le \int_A g$
3. $\forall B \subseteq A$ measurable, $\chi_B f$ is nonnegative measurable and $\int_A \chi_B f = \int_B f$
4. $\forall A_1, A_2 \subseteq A$ disjoint measurable $\int_{A_1\cup A_2} f = \int_{A_1} f + \int_{A_2} f$. If moreover $m(A_2) = 0$, then $\int_{A_2} f = 0$ and so $\int_{A_1\cup A_2} f = \int_{A_1} f$

Proof. 1. $\forall \alpha, \beta > 0$, $\alpha f + \beta g$ is nonnegative measurable since f and g are nonnegative and measurable.

$$\int_A \alpha f = \sup\left\{\int_B h : B \subseteq A \text{ measurable }, m(B) < \infty, h : B \to \mathbb{R} \text{ bd measurable }, 0 \le h \le \alpha f \text{ on } B\right\}$$

This is equivalent to $\alpha \int_B \tilde{h}, \tilde{h} = \frac{h}{\alpha}$ with $0 \le \tilde{h} \le f$ on B

$$= \alpha \sup \left\{ \int_B f : \tilde{h} \subseteq A \text{ measurable }, m(B) < \infty, \tilde{h} : B \to \mathbb{R} \text{ bd measurable }, 0 \le \tilde{h} \le f \text{ on } B \right\}$$

$$= \alpha \int_A f$$

Then, for the sum, we have that:

$$\int_A (f+g) = \sup \left\{ \int_B h : B \subseteq A \text{ measurable }, m(B) < \infty, h : B \to \mathbb{R} \text{ bd measurable }, 0 \le h \le f + g \text{ on } B \right\}$$

$$\ge \sup \left\{ \int_B h_1 : B \subseteq A \text{ measurable }, m(B) < \infty, h_1 : B \to \mathbb{R} \text{ bd measurable }, 0 \le h_1 \le f \text{ on } B \right\}$$

$$+ \sup \left\{ \int_B h_2 : B \subseteq A \text{ measurable }, m(B) < \infty, h_2 : B \to \mathbb{R} \text{ bd measurable }, 0 \le h_2 \le g \text{ on } B \right\}$$

$$= \int_A f + \int_A g$$

Where the inequality arises from the first set containing the second.

Conversely, let B be measurable , $m(B) < \infty, h : B \to \mathbb{R}$ be bounded measurable such that $0 \le h \le f + g$ on B. To show the supremum of these elements is less than the sum of the integrals, we need to show that each of them is less than the sum of the integrals.

Let $h_1 = \min(h, f)$ and $h_2 = h - h_1$ so that $0 \le h_1 \le f$ and $0 \le h_2 = \max(0, h - f) \le g$ on B giving us that they are both bounded measurable.

Therefore, $\int_B h_1 \le \int_A f, \int_B h_2 \le \int_A g$ and so $\int_B h_1 + \int_B h_2 \le \int_A f + \int_A g$.

By taking the supremum, we have that $\int_A (f + g) \le \int_A f + \int_A g$ which combined with the other inequality gives us:

$$\int_A (f + g) = \int_A f + \int_A g$$

2. $f \le g$ on $A \implies \forall B \subseteq A, h : B \to \mathbb{R}[h \le f \implies h \le g]$

$$\implies \int_A f \le \int_A g$$

Since we are taking the supremum of a larger set.

3. If f finite everywhere, then $\forall B \subseteq A$ measurable, $\chi_B f$ is nonnegative and measurable as the product of nonnegative and measurable functions.

$$\int_A \chi_B f = \sup\left\{\int_B h : B' \subseteq A \text{ mbl}, m(B') < \infty, h : B' \to \mathbb{R} \text{ bd mbl}, 0 \le h \le \chi_B f \text{ on } B'\right\}$$

$$\iff 0 \le h \le f \text{ on } B' \cap B, h = 0 \text{ on } B' - B$$

$$\implies \int_B h = \int_{B\cap B'} h + \int_{B-B'} h = \int_{B\cap B'} h$$

Thus, letting $\tilde{h} = h|_{\tilde{B}}$ where $\tilde{B} = B \cap B'$, the supremum is equal to:

$$= \sup\left\{\int_{\tilde{B}} \tilde{h} : \tilde{B} \subseteq B \text{ measurable}, m(\tilde{B}) < \infty, h : \tilde{B} \to \mathbb{R} \text{ bd measurable}, 0 \le \tilde{h} \le f \text{ on } \tilde{B}\right\}$$

4. $\forall A_1, A_2 \subseteq A$ measurable disjoint, we have that (using 3. and 1.):

$$\int_{A_1\cup A_2} f = \int_A f\chi_{A_1\cup A_2} = \int_A f\chi_{A_1} + f\chi_{A_2} = \int_A f\chi_{A_1} + \int_A f\chi_{A_2} = \int_{A_1} f + \int_{A_2} f$$

However to use 3., we need that $|f| < \infty$.

Moreover, once again, if $m(A_2) = 0$, we have $\int_B h = 0$ for all $B \subseteq A_2$ measurable, $m(B) < \infty$, $h : B \to \mathbb{R}$ bd measurable, therefore $\int_{A_2} f = 0$.

□

Theorem 3.7. Chebyshev's Inequality: Let f be measurable nonnegative on $A \subseteq \mathbb{R}$. Then:

$$\forall \lambda > 0 : m(f^{-1}([\lambda, \infty])) \le \frac{1}{\lambda}\int_A f$$

Proof. Define $E_\lambda = f^{-1}([\lambda, \infty])$. Then $f \ge \lambda$ on E_λ and $f \ge 0$ on $A - E_\lambda$.

Thus, $f \ge \lambda \chi_{E_\lambda}$ on A. It follows that $\int_A f \ge \lambda \int_A \chi_{E_\lambda} = \lambda m(E_\lambda)$ □

Corollary 3.8. Let f be nonnegative measurable on A. Then $f = 0$ a.e. in $A \iff \int_A f = 0$

Proof. Chebyshev's Inequality gives that $\forall n \in \mathbb{N} : m(f^{-1}([\frac{1}{n}, \infty])) \le n \int_A f$.

Therefore, if $\int_A f = 0$, then $m(f^{-1}([\frac{1}{n}, \infty])) = 0$. By continuity, we obtain, $m(f^{-1}((0, \infty))) = \lim_{n\to\infty} m(f^{-1}([\frac{1}{n}, \infty])) = 0$. Hence $f = 0$ a.e. in A.

For the other side, assume that $f = 0$ a.e. in A. Let $N = \{x \in A : f(x) \neq 0\}$. Then $m(N) = 0$ and so $\int_N f = 0$. On the other hand, $\int_{A-N} f = \int_{A-N} 0 = 0$. Therefore:

$$\int_A f = \int_N f + \int_{A-N} f = 0$$

□

Corollary 3.9. Let f be nonnegative measurable on A. If f is integrable over A, then $f < \infty$ a.e. in A.

Proof. Chebyshev's Inequality gives that $\forall n \in \mathbb{N} : m(f^{-1}([n,\infty])) \leq \frac{1}{n} \int_A f$ where the RHS goes to 0 as $n \to \infty$.

By continuity, it follows that $m(f^{-1}(\infty)) = \lim_{n\to\infty} m(f^{-1}([n,\infty])) = 0$ □

Lemma 3.10. Fatou's Lemma: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable nonnegative functions on $A \subseteq \mathbb{R}$. Then, $\liminf_{n\to\infty} f_n$ is measurable nonnegative and:

$$\int_A \liminf_{n\to\infty} f_n \leq \liminf_{n\to\infty} \int_A f_n$$

In particular, if $(\int_A f_n)_{n\in\mathbb{N}}$ is bounded by $M < \infty$, then $\liminf_{n\to\infty} f_n$ is integrable and $\int_A \liminf \leq M$.

Proof. Let $f = \liminf_{n\to\infty} f_n$. Let $B \subseteq A$ measurable with $m(B) < \infty$ and $h : B \to \mathbb{R}$ be bounded measurable such that $0 \leq h \leq f$ on B.

Let $h_n = \min(h, \inf_{k\geq n} f_k)$. Then, h_n is measurable, nonnegative, $\sup_B h_n \leq \sup_B h < \infty$ (uniformly bounded), $\lim_{n\to\infty} h_n(x) = \min(h(x), f(x)) = h(x)$ (pointwise convergence).

Thus, we can apply the Bounded Convergence Theorem and get:

$$\int_B h = \lim_{n\to\infty} \int_B h_n$$

Since $h_n \leq f_n$, we have $\int_B h_n \leq \int_B f_n \leq \int_A f_n$ (from nonnegativity) and so $\int_B h \leq \liminf_{n\to\infty} \int_A f_n$.

Finally, taking the supremum yields:

$$\int_A f \leq \liminf_{n\to\infty} \int_A f_n$$

□

Example 3.4. Take $f_n = n\chi_{(0,\frac{1}{n})}$. The integral is 1 $\forall n \in \mathbb{N}$ while the LHS would be 0 since $\int_A \liminf_{n\to\infty} f_n = \int_A 0 = 0$.

Theorem 3.11. Monotone Convergence Theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on $A \subseteq \mathbb{R}$ such that $\forall n \in \mathbb{N}: f_n \leq f_{n+1}$

Since the sequence is increasing, we have that $\lim_{n\to\infty} f(x)$ exists in $[0,\infty]$ for all $x \in A$ and $\lim_{n\to\infty} \int_A f$ exists in $[0,\infty]$.

Then, we have that:

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n$$

Proof. By Fatou's Lemma, we have:

$$\int_A \lim_{n\to\infty} f_n = \int_A \liminf_{n\to\infty} f_n = \leq \liminf \int_A f_n = \lim_{n\to\infty} \int_A f_n$$

Since $f_n \leq f_{n+1} \forall n \in \mathbb{N}$, we have $f_n \leq \lim_{k\to\infty} f_k$. Hence $\int_A f_n \leq \int_A \lim_{k\to\infty} f_k$ and so

$$\int_A \lim_{n\to\infty} f_n = \lim_{n\to\infty} \int_A f_n$$

□

Corollary 3.12. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative measurable functions on $A \subseteq \mathbb{R}$. Then:

$$\int_A \sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} \int_A u_n$$

Proof. Apply the Monotone Converge theorem to the partial sums $\sum_{n=1}^{N} u_n$ □

3.4 Case of possibly sign-changing functions

Def. We say that a measurable function $f : A \to \overline{\mathbb{R}}$ is **integrable** over A if $f_+ = \max(f,0)$ and $f_- = \max(-f,0)$ are integrable.

We then denote:

$$\int_A f = \int_A f_+ - \int_A f_-$$

For every $B \subseteq A$ measurable, $\int_B f = \int_B f|_B$.

Proposition 3.13. f is integrable iff $|f|$ is integrable.

Proof. Follows from the fact that $|f| = f_+ + f_-$ □

Proposition 3.14. Let f, g be integrable over $A \subseteq \mathbb{R}$:

1. $\forall \alpha, \beta \in \mathbb{R} : \alpha f + \beta g$ is integrable $\int_A \alpha f + \beta g = \alpha \int_A f + \beta \int_A g$
2. If $f \leq g$ on A, then $\int_A f \leq \int_A g$
3. $|\int_A f| \leq \int_A |f|$
4. $\forall B \subseteq A$ measurable, $f\chi_B$ is integrable and $\int_A f\chi_B = \int_B f$
5. $\forall A_1, A_2 \subseteq A$ measurable disjoint, $\int_{A_1 \cup A_2} = \int_{A_1} f + \int_{A_2} f$. If moreover $m(A_2) = 0$, then $\int_{A_2} f = 0$ and so $\int_{A_1 \cup A_2} = \int_{A_1} f$

Proof. Straightforward □

Remark. We need to watch out for the following two cases:

- $f + g$ is not well defined in the set $N = \{x \in A : f(x) = -g(x) \in \{\pm\infty\}\}$ but if f and g are integrable, then $|f|, |g| < \infty$ a.e. in A and so $m(N) = 0$. In this case, we still say that $f + g$ is integrable over A and we denote $\int_A (f + g) = \int_{A-N} (f + g)$.
- Same for $f\chi_B : \int_A f\chi_B = \int_{A-N} f\chi_B, N = \{x \in A : |f(x)| = \infty\}$ if f is integrable.

Theorem 3.15. Dominated Convergence Theorem: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of measurable functions on $A \subseteq \mathbb{R}$ such that:

1. $\exists g$ integrable over A such that $\forall n \in \mathbb{N} : |f_n| \leq g$ a.e. on A.
2. $\exists f : A \to \overline{R}$ such that $f_n \to f$ pointwise a.e. in A.

Then the functions f_n and f are integrable and:

$$\int_A f = \lim_{n\to\infty} \int_A f_n$$

Proof. Since 1. and 2. are true a.e. in A, there exists $A' \subseteq A$ such that $m(A - A') = 0$ and both properties hold everywhere on A'.

Since $|f_n| \leq g$ and g is integrable, so is $f_n, \forall n \in \mathbb{N}$. Then, considering both properties on A', we have that $|f| \leq g$ on A' so f is integrable.

Let $g_n^+ = g + f_n$ and $g_n^- = g - f_n$ so that both are nonnegative and measurable (as sum/difference of measurable functions). By Fatou's Lemma, we have:

$$\int_A \liminf g_n^\pm \leq \liminf_{n\to\infty} \int g_n^\pm$$

Where the LHS is $\int_A g \pm f$ and the RHS is $\liminf_{n\to\infty} \int_A g \pm \int_A f_n$.

Then, by subtracting $\int_A g$, we have:

$$\pm \int_A f \leq \liminf_{n\to\infty} (\pm \int_A f_n)$$

and so

$$\int_A \leq \liminf_{n\to\infty}\int_A f_n$$
$$\int_A f \geq -\liminf_{n\to\infty} - \int_A f_n = \limsup_{n\to\infty}\int_A f_n$$

which gives

$$\lim_{n\to\infty}\int_A f_n = \int_A f$$

□

3.5 Applications of the Dominated Convergence Theorem.

Corollary 3.16. Countable additivity of integration: Let f be integrable over $A \subseteq \mathbb{R}$ and $(A_n)_{n\in\mathbb{N}}$ be measurable and disjoint subsets of A. Then:

$$\int_{\cup_{n=1}^{\infty} A_n} f = \sum_{n=1}^{\infty}\int_{A_n} f$$

Proof. Let $f_n = f\chi_{\cup_{k=1}^{n} A_k}$. Then f_n is measurable, $|f_n| \leq |f|$ on $\bigcup_{k=1}^{\infty} A_K$ and $\forall x \in \bigcup_{k=1}^{\infty} A_K, \exists n_x \in N : \forall n \geq n_x, x \in \bigcup_{k=1}^{n} A_k$ (take N_x such that $x \in A_{n_x}$)) and then $f_n(x) = f(x)$ (i.e. we have pointwise convergence of f_n).

Since f is integrable, it follows from the DCT that f_n are integrable and

$$\int_{\cup_{k=1}^{\infty} A_k} f = \lim_{n\to\infty}\int_{\cup_{k=1}^{\infty} A_k} f_n = \lim_{n\to\infty}\int_{\cup_{k=1}^{n}} f = \sum_{k=1}^{\infty}\int_{A_K} f$$

□

Corollary 3.17. Continuity of integration: Let f be integrable over $A \subseteq \mathbb{R}$ and $(A_n)_{n\in\mathbb{N}}$ be measurable subsets of A. Then:

1. If $A_n \subseteq A_{n+1} : \forall n \in N$, then $\int_{\bigcup_{k=1}^{\infty}} = \lim_{n\to\infty}\int_{A_n} f$
2. If $A_{n+1} \subseteq A_n : \forall n \in N$, then $\int_{\bigcap_{k=1}^{\infty}} = \lim_{n\to\infty}\int_{A_n} f$

No need to worry about unbounded integral since bounded by finite integral.

Proof. Exercise in assignment 4. □

Example 3.5. If f is integrable over $\mathbb{R}$, then $\lim_{n\to\infty}\int_{[-n,n]} f = \int_{\mathbb{R}} f$.

4 Differentiation and Integration

Problem 1: Given $f : [a,b] \to \overline{\mathbb{R}}$ integrable, is the function $F(x) = \int_a^x f(t)d(t)$ differentiable and is $F'(x) = f(x)$, at least a.e. in $[a,b]$.

Problem 2: Under which conditions on a function $F : [a,b] \to \mathbb{R}$ does there exists $f : [a,b] \to \overline{\mathbb{R}}$ integrable, such that $F(x) = \int_a^x f(t)d(t) + F(A), \forall x \in [a,b]$.

We first consider the case of **monotone functions** (not necessarily continuous)

Theorem 4.1. Every monotone function $f : [a,b] \to \mathbb{R}$ is differentiable a.e. in $[a,b]$. Furthermore, f' is integrable over $[a,b]$ and:

- If f is increasing, then $\int_a^b f' \leq f(b) - f(a)$
- If f is decreasing, then $\int_a^b f' \geq f(b) - f(a)$

Proof. WLOG, we may assume that f is increasing (otherwise consider $-f$ if it is decreasing).

We have $\forall x \in (a,b)$, f is not differentiable at $x, \underline{D}f(x) < \infty \iff \underline{D}f(x) < \overline{D}f(x)$ (and thus the limit does not exist) where

- $\underline{D}f(x) = \liminf_{t\to 0, t\neq 0} \frac{f(x+t)-f(x)}{t}$
- $\overline{D}f(x) = \limsup_{t\to 0, t\neq 0} \frac{f(x+t)-f(x)}{t}$

Since f is increasing, we have $\underline{D}f(x) \geq 0$ and $\overline{D}f(x) \geq 0$.

$$\underline{D}f(x) < \overline{D}f(x) \iff \exists \alpha, \beta \in \mathbb{R} : \underline{D}f(x) < \alpha < \beta < \overline{D}f(x) \iff \exists \alpha, \beta \in \mathbb{Q} : \underline{D}f(x) < \alpha < \beta < \overline{D}f(x)$$

It follows that

$$\left\{x \in [a,b] : \underline{D}f(x) < \overline{D}f(x)\right\} = \bigcup_{\alpha,\beta\in\mathbb{Q}, 0<\alpha<\beta} A_{\alpha,\beta}$$

where

$$A_{\alpha,\beta} = \left\{x \in [a,b] : \underline{D}f(x) < \alpha < \beta < \overline{D}f(x)\right\}$$

Therefore, to prove that $m(\{x \in [a,b] : \underline{D}f(x) < \overline{D}f(x)\}) = 0$, it suffices to prove that $m(A_{\alpha,\beta}) = 0, \forall \alpha, \beta \in \mathbb{Q}, 0 < \alpha < \beta$.

Fix α, β and let $\epsilon > 0$. Since $A_{\alpha,\beta} \subseteq (a,b), m^*(A_{\alpha,\beta}) < l((a,b)) = b - a$ and so $\exists O_\epsilon \subseteq (a,b)$ open (union of countable open bounded intervals) such that $A_{\alpha,\beta} \subseteq O_\epsilon$ and $m^*(O_\epsilon) < m^*(A_{\alpha,\beta}) + \epsilon$.

Step 1: Show that $\exists [a_1,b_1],[a_2,b_2]...[a_n,b_n]$ closed bounded disjoint intervals such that $\bigcup_{k=1}^n [a_k,b_k] \subseteq O_\epsilon, m^*(A_{\alpha,\beta} - \bigcup_{k=1}^n [a_k,b_k]) < \epsilon$ and $\forall k \in \{1...n\} : \frac{f(b_k)-f(a_k)}{b_k-a_k} < \alpha$

Let $F = \{[a',b']\} \subset O_\epsilon : \frac{f(b')-f(a')}{b'-a'} < \alpha$

Remark. $\forall x \in A_{\alpha,\beta}, \forall \epsilon > 0$, since $\underline{D}f(x) < \alpha, \exists I \in F$ such that $x \in I$ and $l(I) < \epsilon$. A covering F of $A_{\alpha,\beta}$ satisfying the above is called a **Vitaliy covering**.

We choose the intervals $I_k = [a_k, b_k]$ by induction so that $I_{k+1} \in F_k$ and $\forall k \in N : l(I_{k+1}) > \frac{1}{2}\sup_{I\in F_k} l(I)$ where $F_k = \{I \in F : I \cap \cup_{j=1}^{k} I_j = \emptyset\}$, provided $F_k \neq \emptyset$.

Case 1: $\exists n \in \mathbb{N} : F_n = \emptyset$. In this case, the induction stops. Moreover, since $F_n = \emptyset, \forall I \in F : I \cup \bigcup_{k=1}^{n} I_k \neq \emptyset$. In this case, we will show that $A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k = \emptyset$.

Assume for contradiction that $\exists x \in A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k$. Since $\bigcup_{k=1}^{n} I_k$ is closed (and thus complement is open), $\exists \delta_x > 0$ such that $(x-\delta_x, x+\delta_x) \cap \bigcup_{k=1}^{n} I_k = \emptyset$. Moreover, by the Vitali property, we have that $\exists I \subset (x-\delta_x, x+\delta_x)$ such that $I \in F$ and so $I \in F_n$. ↯

Case 2: $\forall n \in \mathbb{N} : F_n \neq \emptyset$: We will show that $m^*(A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k) < \epsilon$ for some $n_\epsilon \in \mathbb{N}$. Let $n \in \mathbb{N}$ and $x \in A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k$.

As in the previous case, $\exists I \in F_n$ such that $x \in I_X$. If $I_X \cap I_K = \emptyset, \forall k > n$, then we have that $I_x \in F_k, \forall k \in \mathbb{N}$ and so $l(I_k) > \frac{1}{2} l(I_x)$ which implies $l(\cup_{k=n+1}^{\infty} I_k) = \infty$, contradicts $(I_k)_{k\geq n}$ disjoint included in (a, b) bounded, ↯.

It follows that, $\exists k_x > n$ (take first one) such that $I_x \in F_{k_x - 1}$ and $I_x \cap I_{k_x} \neq \emptyset$. Let y_{k_x} be the middle point of I_{k_x}. Let $z_k \in I_x \cap I_{k_x}$. We have $|x - y_{k_x}| \leq |x - z_x| + |z_x - y_{k_x} < l(I_x) + \frac{1}{2} l(I_{k_x})$. Since $l(I_{k_x}) > \frac{1}{2} l(I_x)$, it follows that $|x - y_{k_x}| < 2l(I_{k_x}) + \frac{1}{2} l(I_{k_x}) = \frac{5}{2} l(I_{k_x})$ and so $x \in [y_{k_x} - \frac{5}{2} l(I_{k_x}), y_{k_x} + \frac{5}{2} l(I_{k_x})] = I'_{k_x}$ with length $\frac{5}{2} l(I_{k_x})$.

We have proven that $A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k \subseteq \bigcup_{k=n+1}^{\infty} I'_k$. It follows that $m^*(A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k) \leq \sum_{k=n+1}^{\infty} l(I'_k) = 5\sum_{k=n+1}^{\infty} l(I_k)$. Since $\sum_{k=1}^{\infty} I_k = l(\bigcup_{k=1} \infty I_k) \leq l((a, b)) < \infty$, we have that $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} l(I_k) = 0$ and so $\exists n_\epsilon \in \mathbb{N} : m^*(A_{\alpha,\beta} - \bigcup_{k=1}^{n} I_k) < \epsilon$

Step 2: Show that $\forall k \in \{1...N\} : \exists [a_{k,1}, b_{k,1}]...[a_{k,N}, b_{k,N}] \subseteq (a_k, b_k)$ such that $m^*(A_{\alpha,\beta} \cap (a_k, b_k) - \bigcup_{j=1}^{N_k} [a_{k,j}, b_{k,j}]) < \epsilon$ and $\forall j \in \{1...N_k\} : \frac{f(b_{k,j}) - f(a_{k,j})}{b_{k,j} - a_{k,j}} > \beta$

Same proof as Step 1 with the Vitali covering $F = \{[a', b'] \subset (a_k, b_k) : \frac{f(b') - f(a')}{b' - a'} > \beta\}$

Step 3: Show that $m^*(A_{\alpha,\beta}) = 0$ which completes the proof.

From Steps 1 and 2, we have:

$$\sum_{k=1}^{N} (f(b_k) - f(a_k)) < \alpha \sum_{k=1}^{N} (b_k - a_k)$$

Since the function is increasing, we have that

$$\sum_{k=1}^{N} \sum_{k=1}^{N} (f(b_{k,j}) - f(a_{k,j})) < \alpha \sum_{k=1}^{N} (b_k - a_k)$$

$$\beta \sum_{k=1}^{N} \sum_{j=1}^{N_k} (b_{k,j} - a_{k,j}) < \sum_{k=1}^{N} \sum_{k=1}^{N} (f(b_{k,j}) - f(a_{k,j})) < \sum_{k=1}^{N} f(b_k) - f(a_k) < \alpha \sum_{k=1}^{N} (b_k - a_k) < \alpha m^*(O_\epsilon) \leq \alpha(m^*(A_{\alpha,\beta}) + \epsilon)$$

Where the first inequality is from Step 2, the second from f increasing, the third from Step 3, the fourth since the a_k, b_k are disjoint and contained in O_ϵ and the last from the measure of O_ϵ being ϵ close.

Thus, we have that:

$$m^*(A_{\alpha,\beta}) \leq m^*(A_{\alpha,\beta} \bigcap \bigcup_{k=1}^{N} [a_k, b_k]) + m^*(A_{\alpha,\beta} - \bigcup_{k=1}^{N} [a_k, b_k])$$

Since the second term is $< \epsilon$ by Step 1 and using subadditivity:

$$m^*(A_{\alpha,\beta}) \leq \sum_{k=1}^{N} m^*(A_{\alpha,\beta} \cap [a_k, b_k]) + \epsilon$$

$$\leq \sum_{k=1}^{N} (m^*(A_{\alpha,\beta} \cap [a_k, b_k] \cap \cup_{j=1}^{N_k} [a_{k,j}, b_{k,j}]) + m^*(A_{\alpha,\beta} \cap [a_k, b_k] - \cup_{j=1}^{N_k} [a_{k,j}, b_{k,j}])) + \epsilon$$

The second term is $< \epsilon$ by Step 2.

$$\leq \sum_{k=1}^{N} (\sum_{j=1}^{N_k} m^*(A_{\alpha,\beta} \cap [a_{k,j}, b_{k,j}]) + \epsilon) + \epsilon$$

$$\leq \sum_{k=1}^{N} (\sum_{j=1}^{N_k} (b_{k,j} - a_{k,j} + \epsilon) + \epsilon) < \frac{\alpha}{\beta}(m^*(A_{\alpha,\beta}) + \epsilon) + N\epsilon + \epsilon$$

Letting $\epsilon \to 0$, we obtain that $m^*(A_{\alpha,\beta}) \leq \frac{\alpha}{\beta} m^*(A_{\alpha,\beta})$ and so $m^*(A_{\alpha,\beta}) = 0$ since $\frac{\alpha}{\beta} < 1$

Step 4: Show that f is differentiable a.e. in (a, b) and $\int_{[a,b]} f' \leq f(b) - f(a)$.

Let $D_{\frac{1}{n}} f(x) = \frac{f(x+\frac{1}{n})-f(x)}{\frac{1}{n}}$ if $x \in [a, b - \frac{1}{n}]$ and 0 if $x \in (b - \frac{1}{n}, b]$.

- $D_{\frac{1}{n}} f \geq 0$ since f is increasing.
- $D_{\frac{1}{n}} f \geq 0$ is measurable. Suffices to show that f is measurable. f is measurable as an increasing function since $\forall c \in \mathbb{R} : f^{-1}((-\infty, c))$ is an interval. Indeed $\forall y < z \in f^{-1}((-\infty, c)) :$ $\forall x \in (y, z) : f(x) \leq f(z) < c$ so $x \in f^{-1}((-\infty, c))$
- For a.e. in $x \in (a, b) : (D_{\frac{1}{n}} f(x))_{n \in \mathbb{N}}$ converges to $\underline{D} f(x) = \overline{D} f(x) = Df(x) \in [0, \infty]$. Follows from Step 3.

Fatou's Lemma gives $\int_{[a,b]} Df \leq \liminf_{n\to\infty} \int_{[a,b]} D_{\frac{1}{n}} f$. Then:

$$\int_{(a,b)} D_{\frac{1}{n}} = n(\int_{(a,b-\frac{1}{n})} f(x + \frac{1}{n}) dx + \int_{(a,b-\frac{1}{n})} f(x) dx)$$

The first term is $= \int_{(a+\frac{1}{n},b)} f(x) dx$ by translation invariance of the measure (left as exercise).

$$= n(\int_{(b-\frac{1}{n},b)} f(x) dx - \int_{(a,a+\frac{1}{n})} f(x) dx)$$

Since $f(a) \le f(x) \le f(b)$:

$$\le n(f(b) - m((b-\frac{1}{n}, b)) - f(a)m((a, a+\frac{1}{n}))) = f(b) - f(a)$$

So $\int_{(a,b)} Df \le f(b) - f(a)$. In particular, $Df < \infty$ a.e. and so f' exists a.e.

□

Example 4.1. The Cantor-Lebesgue function $\phi : [0,1] \to [0,1]$ is increasing but $\phi' = 0$ a.e. in $[0,1]$ since it is constant a.e. and so $\int_0^1 \phi' = 0 < 1 = \phi(1) - \phi(0)$

4.1 Functions of bounded variation

Def. We say that a function $f : [a,b] \to \mathbb{R}$ is of **bounded variation** if $TV(f) < \infty$ where:

$$TV(f) = \sup\left\{\sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k)|\right\}; a = x_0 < x_1 < ... < x_N = b$$

$TV(f)$ is called the **total variation** of f.

Example 4.2. Some examples of functions of bounded variation:

1. Monotone functions: If $f : [a,b] \to \mathbb{R}$ is increasing, then for all partitions of $[a,b]$:

$$TV(f) = \sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k)| = f(b) - f(a)$$

   Since other terms cancel. Thus, the TV is finite. Similarly, if f is decreasing, $TV(f) = f(a) - f(b)$.

2. Lipschitz functions: If $\forall x, y \in [a,b] : |f(y) - f(x)| \le C|y-x|$. Then, for any partition:

$$\sum_{k=0}^{N-1} |f(x_{k+1} - f(x_k)| \le c \sum_{k=0}^{N-1} |x_{k+1} - x_k| = C(b-a)$$

   And thus f is of bounded variation and $TV(f) \le C(b-a)$

Proposition 4.2. Let $f : [a,b] \to \mathbb{R}$ and $c \in (a,b)$. Then:

$$TV(f) = TV(f|_{[a,c]}) + TV(f|_{[c,b]})$$

Proof. $\forall a = x_0 < x_{..} < x_N = c, c = y_0 < ... < y_{N'} = b$

$$\sum_{k=0}^{N-1} |f(x_{k+1}, x_k) + \sum_{k=0}^{N'-1} |f(y_{k+1}) - f(y_k)| \le TV(f)$$

Since combining them yields a partition of the whole interval.

Hence, $TV(f|_{[a,c]}) + TV(f|_{[c,b]}) \leq TV(f)$

Conversely, $\forall a = x_0 < ... x_N = b$, letting $i_0 \in \{0, ... N-1\}$ such that $x_{i_0} \leq c < x_{i_0+1}$:

$$\sum_{k=0}^{N-1} |f(x_{k+1}) - f(x_k)| = \sum_{k=0}^{i_0-1} |f(x_{k+1}) - f(x_k)| + |f(x_{i_0+1}) - f(x_{i_0})| + \sum_{k=i_0}^{N-1} |f(x_{k+1}) - f(x_k)|$$

$$\leq \sum_{k=0}^{i_0-1} |f(x_{k+1}) - f(x_k)| + |f(x_{i_0+1}) - f(c)| + |f(c) - f(x_{i_0})| + \sum_{k=i_0}^{N-1} |f(x_{k+1}) - f(x_k)|$$

$$\leq TV(f|_{[a,c]}) + TV(f|_{[c,b]})$$

□

Theorem 4.3. A function $f : [a,b] \to \mathbb{R}$ is of bounded variation iff it can be written as the difference as the difference of two increasing functions.

In particular, every function of bounded variation is differentiable a.e. in $[a,b]$ and its derivative is integrable over $[a,b]$.

Proof. Assume f is of bounded variation. Then $f(x) = (f(x) + TV(f|_{[a,x]})) - TV(f|_{[a,x]})$. We now check they are both increasing. $\forall x < y \in [a,b]$:

$$TV(f|_{[a,y]}) - TV(f|_{[a,x]}) = TV(f|_{[x,y]}) \geq 0$$

and

$$(f(y) + TV(f|_{[a,y]})) - (f(x) + TV(f|_{[a,x]})) = (f(y) - f(x)) + TV(f|_{[x,y]}) > -|f(y) - f(x)| + TV(f|_{[x,y]}) \geq 0$$

Thus, both functions are increasing.

Assume $f = g - h$ where g and h are moving. Then g and $-h$ are of bounded variations and so their sum f is of bounded variation (see A5Q4). □

4.2 Absolutely continuous functions

We say that a function $f : [a,b] \to \mathbb{R}$ is **absolutely continuous** if $\forall \epsilon > 0 : \exists \delta_\epsilon > 0$: such that for all finite collections of disjoint open bounded intervals $(a_1, b_1) ... (a_N, b_N)$, if $\sum_{k=1}^{N} (b_k - a_k) < \delta_\epsilon$, then $\sum_{k=1}^{N} |f(b_k) - f(a_k)| < \epsilon$.

Clearly, absolute continuity $\implies$ uniform continuity by taking $N = 1$.

Example 4.3. Consider the following functions:

1. Lipschitz functions are absolutely functions.

   If $\forall x, y \in [a,b] : |f(x) - f(y)| \leq C|x-y|$, then $\forall \epsilon > 0, \forall (a_1, b_1)...(a_N, b_N)$ open bounded disjoint intervals, if $\sum_{k=1}^{N}(b_k - a_k) < \delta_\epsilon = \frac{\epsilon}{C}$, then $\sum_{k=1}^{N}|f(b_k) - f(a_k)| \leq C\sum_{k=1}^{N}|b_k - a_k| < \epsilon$

2. Cantor-Lebesgue function is not absolutely continuous.

   Let $C = \bigcap_{k=1}^{\infty} C_k, C_k = \bigcap_{j=1}^{2^k}[a_{k,j}, b_{k,j}]$ where $[a_{k,j}, b_{k,j}]$ are disjoint intervals of length 3^{-k}. Then $\sum_{j=1}^{2^k}(b_{k,j} - a_{k,j} = \frac{2^k}{3^k}) \to 0$ as $k \to \infty$.

   On the other hand, by the definition of the Cantor-Lebesgue function ϕ, we have that $\sum_{j=1}^{2^k}|\phi(b_{k,j}) - \phi(a_{k,j})| = \sum_{j=1}^{2^k} 2^{-k} = 1^k = 1$ and thus ϕ is not absolutely continuous.

Theorem 4.4. Every absolutely continuous function $f : [a,b] \to \mathbb{R}$ can be written as the difference of two increasing absolutely continuous functions. In particular, it is of bounded variation.

Proof. Multiple steps:

Step 1: Show that $\forall \epsilon > 0 : \exists \delta_\epsilon > 0 : \forall (a_1, b_1)...(a_N, b_N)$ open bounded disjoint, if $\sum_{k=1}^{N}(b_k - a_k) < \delta_\epsilon$, then $\sum_{k=1}^{N} TV(f|_{[a_k,b_k]}) < \epsilon$

Let $\epsilon > 0$. Since f is absolutely continuous, $\exists : \delta_\epsilon : \forall (a_1', b_1')...(a_N', b_N')$ disjoint open bounded, if $\sum_{k=1}^{N} b_k' - a_k' < \delta_\epsilon$ then $\sum_{k=1}^{N} f(b_k') - f(a_k') < \epsilon$.

Then $\forall (a_1, b_1)...(a_k, b_k)$ disjoint, open bounded $\forall a_k = x_{k,0} < x_{k,1} ... < x_{k,N_k} = b_k$, if

$$\sum_{k=1}^{N}\sum_{j=1}^{N_k - 1} x_{k,j+1} - x_{k,j} = \sum_{k=1}^{N} b_k - a_k < \delta_\epsilon$$

Then:

$$\sum_{k=1}^{N}\sum_{j=1}^{N_k - 1} |f(x_{k,j+1}) - f(x_{k,j})| < \epsilon$$

By taking the supremum, we obtain $\sum_{k=1}^{N} TV(f|_{[a_k,b_k]}) < \epsilon$.

Step 2: f is of bounded variation. We have:

$$TV(f) = \sum_{k=1=1}^{n} TV(f|_{I_k = [a + \frac{b-a}{n}(k-1), a + \frac{b-a}{n}k]})$$

Choosing n such that $\frac{b-a}{n} < \delta_1$, we obtain that $TV(f|_{I_k}) < 1$ so $TV(f) < n$ and is thus finite.

Step 3: Show that $x \to TV(f|_[a, x])$ is absolutely continuous. Remark that $|TV(f|_{[a,b_k]}) - TV(f|_{[a,a_1]})| = TV(f|_{[a_k,b_k]})$ so the result follows directly from Step 1.

Write $f(x) = (f(x)+TV(f|_{[a,x]}))-TV(f|_{[a,x]})$. We have that both functions are absolutely continuous using that the sum of 2 absolutely continuous functions is absolutely continuous (A5Q4). Moreover, we proved that the functions are increasing. □

Theorem 4.5. Let $f : [a,b] \to \mathbb{R}$:

1. If f is absolutely continuous on $[a,b]$, then $\forall x \in [a,b] \int_{[a,x]} f' = f(x) - f(a)$
2. Conversely, if $\exists g$ integrable over $[a,b]$ such that $\forall x \in [a,b] : \int_{[a,x]} g = f(x) - f(a)$, then f is absolutely continuous and $f' = g$ almost everywhere in $[a,b]$.

Proof. WLOG, we may assume that f is increasing. By continuity of f and $x \to \int_{(a,x)} f'$, WLOG we may assume that $x < b$. We let:

$$D_{\frac{1}{n}} f(y) = \begin{cases} \frac{f(y+\frac{1}{n})-f(y)}{\frac{1}{n}} & \text{if } y \in [a, b-\frac{1}{n}] given a < b - \frac{1}{n} \\ 0 & \text{if } y \in \frac{b-\frac{1}{n}}{b} \end{cases}$$

Then $D_{\frac{1}{n}} f(y) \geq 0$ and measurable.

Step 1: Show that $\lim_{n\to\infty} \int D_{\frac{1}{n}} f = f(x) - f(a)$

$$\int_{[a,x]} D_{\frac{1}{n}} f = n[\int_{[a,x]} [f(y+\frac{1}{n}) - f(y)] dy]$$

$$= n[\int_{[a+\frac{1}{n}, x+\frac{1}{n}]} - \int_{[a,x]} f]$$

provided $x + \frac{1}{n} < b$

$$= n[\int_{[x,x+\frac{1}{n}]} f - \int_{[a,a+\frac{1}{n}]} f]$$

provided $a + \frac{1}{n} < x$

Moreover, $\int_{[x,x+\frac{1}{n}]} f - f(x) = n(\int_{[x,x+\frac{1}{n}]} f(y)dy - \int_{[x,x+\frac{1}{n}]} f(x)dy) = n(\int_{[x,x+\frac{1}{n}]} f(y)dy - f(x) * m([x, x+\frac{1}{n}]))$

$$= n \int_{[x,x+\frac{1}{n}]} f(y) - f(x) dy$$

$$\leq n \sup_{y \in [x,x+\frac{1}{n}]} |f(y) - f(x)| m([x, x+\frac{1}{n}]) = \sup_{y \in [x,x+\frac{1}{n}]} |f(y) - f(x)| \to 0 \text{ as } n \to \infty$$

Step 2: Show that $\lim_{n\to\infty} \int_{[a,x]} D_{\frac{1}{n}} f = \int_{[a,x]} f'$

Since f is differentiable a.e. in $[a,b]$, we have $D_{\frac{1}{n}}f \to f'$ pointwise a.e. in (a,b). By Egoroff's Theorem, $\forall \delta > 0 : \exists F_\delta \subseteq (a,b)$ closed such that $D_{\frac{1}{n}}f \to f'$ uniformly on F_δ and $m((a,b)-F_\delta) < \delta$. Since $(a,b)-F_\delta$ is open,$\exists (a_k,b_k)_{k\in\mathbb{N}}$ disjoint open bounded intervals such that $(a,b)-F_\delta = \bigcup_{k\in\mathbb{N}}(a_k,b_k)$

Then, $\forall N \in \mathbb{N}$:

$$\int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} D_{\frac{1}{n}}f = \sum_{k=1}^{N}\int_{(a_k,b_k)} D_{\frac{1}{n}}f = m\sum_{k=1}^{N}\int_{(a_k,b_k)}((f(x+\frac{1}{n})-f(y))dy$$

provided $x+\frac{1}{n} < b$

$$= n\sum_{k=1}^{N}(\int_{(b_k,b_k+\frac{1}{n})} f - \int_{(a_k,a_k+\frac{1}{n})})$$

$$= n\sum_{k=1}^{N}\int_{(0,\frac{1}{n})} f(b_k+y)-f(a_k+y)dy = n\int_{(0,\frac{1}{n})}\sum_{k=1}^{n}(f(b_k+y)-f(a_k+y))$$

Remark that $\sum_{k=1}^{N}(b_k+y)-(a_k+y) = \sum_{k=1}^{N}(b_k-a_k) = m(\bigcup_{k\in\mathbb{N}}(a_k,b_k)) \to m((a,x)-F_\delta)$ as $N\to\infty$ by continuity of the measure.

Since $m((a,x)-F_\delta) < \delta, \exists N_\delta \in \mathbb{N} : \forall N \geq N_\delta : \forall y \in (0,\frac{1}{n}) : \sum_{k=1}^{N}((b_k+y)-(a_k+y)) < \delta$

Since f is absolutely continuous, $\forall \epsilon > 0 : \exists \delta_\epsilon > 0$ such that if $\sum_{k=1}^{N}((b_k+y)-(a_k+y)) < \delta_\epsilon$ then $\sum_{k=1}^{N}(f(b_k+y)-f(a_k+y)) < \epsilon$

Therefore, $\forall N \geq N_{\delta_\epsilon} : \forall n \in \mathbb{N} : \forall y \in (0,\frac{1}{n}) : \sum_{k=1}^{N}(f(b_k+y)-f(a_k+y)) < \epsilon$ and so $\int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} D_{\frac{1}{n}}f < n\int_{(0,\frac{1}{n})}\epsilon = \epsilon$ for n sufficiently large.

By continuity of integration, we have that $\lim_{N\to\infty}\int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} D_{\frac{1}{n}}f = \int_{(a,x)-F_\delta} D_{\frac{1}{n}}f$

Then by Fatou's Lemma, since $D_{\frac{1}{n}}f \geq 0$ and $D_{\frac{1}{n}}f \to f'$ a.e. pointwise, we have that $\int_{(a,x)-F_\delta} f' \leq \liminf_{n\to\infty}\int_{(a,x)-F_\delta} D_{\frac{1}{n}}f$.

Thus,

$$|\int_{(a,x)}(D_{\frac{1}{n}}f-f')| = \int_{F_\delta}|(D_{\frac{1}{n}}f-f')| + \int_{(a,x)-F_\delta}|(D_{\frac{1}{n}}f-f')|$$

$$\leq m(F_\delta)\sup_{F_\delta}|(D_{\frac{1}{n}}f-f')| + \int_{(a,x)-F_\delta} D_{\frac{1}{n}}f + \int_{(a,x)-F_\delta} f'$$

So

$$\forall \epsilon > 0 : \limsup_{n\to\infty}\int_{(a,x)}|(D_{\frac{1}{n}}f-f')| \leq \limsup_{n\to\infty}\int_{(a,x)-F_\delta} D_{\frac{1}{n}}f + \int_{(a,x)-F_\delta} f' \leq 2\epsilon$$

Letting $\epsilon \to 0$, we obtain $\lim_{n\to\infty} |\int_{(a,x)} D_{\frac{1}{n}} f - \int_{(a,x)} f'| = 0$

Proof of ii) Assume that $\exists g$ integrable over $[a,b]$ such that $\forall x \in [a,b] f(x) - f(a) = \int_{(a,x)} g$. Let $\epsilon > 0$ and $(a_1, b_1)...(a_N, b_N)$ be open, bounded, disjoint in (a,b):

$$\sum_{k=1}^{N} |f(b_k) - f(a_k)| = \sum_{k=1}^{N} |\int_{(a_k,b_k)} g| \leq \sum_{k=1}^{N} \int_{(a_k,b_k)} |g| = \int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} |g|$$

Let $g_j = \min(|g|, j)$ for $j \in \mathbb{N}$. Then, g_j is measurable, $g_{j+1} \geq g_j \geq 0$, $\lim_{j\to\infty} g_J(x) = |g(x)|$ so monotone convergence gives:

$$\int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} g_j \to \int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} |g|$$

In particular, $\forall \epsilon > 0, \exists g_\epsilon \in \mathbb{N}$:

$$\int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} |g| < \int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} |g| < \int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} g_{j_\epsilon} + \frac{\epsilon}{2}$$

Where $|int|g| - \int g_{j_\epsilon}| < \frac{\epsilon}{2}$

$$\leq j_\epsilon m(\bigcup_{k\in\mathbb{N}} (a_k, b_k)) + \frac{\epsilon}{2}$$

Then, if $\sum_{k=1}^{N} (b_k, a_k) < \delta_\epsilon = \frac{\epsilon}{2j_\epsilon} \implies \int_{\bigcup_{k\in\mathbb{N}}(a_k,b_k)} |g| < \frac{\epsilon}{2} + \frac{\epsilon}{2}$

This proves the absolute continuity of f. Then, from 1. it follows that $\int_{(a,x)} f' = f(x) - f(a) = \int_{(a,x)} g$ so $\int_{(a,x)} (f' - g) = 0$ and so $\forall x < y \in (a,b) : \int_{(x,y)} (f' - g) = \int_{(a,y)} (f' - g) - \int_{(a,x)} (f' - g) = 0$. We will conclude that $f' = g$ a.e. in (a,b) by applying:

Lemma 4.6. Let h integrable over $[a,b]$. Then $h = 0$ a.e. in $[a,b] \iff \forall x < y \in (a,b) : \int_{x,y} h = 0$

Proof. $\implies$ is trivial.

$\impliedby$ Assume that $\forall x < y \in [a,b] \int_{(x,y)} h = 0$. Since every open set in (a,b) can be written as a countable union of disjoint intervals, by continuity of integration we obtain that $\forall O \subseteq (a,b)$ open $\int_O h = 0$.

Again, by continuity of integration, it follows that $\int_G h = 0$ for all G_δ set $G \subseteq (a,b)$.

For every measurable $A \subseteq (a,b)$, we have that $A = G - N$ where G is a G_δ set and $m(N) = 0$ so $\int_a h = \int_G h - \int_N h = 0$ since $m(N) = 0$. Then, we conclude, as in Q6 of Assignment 3 □

□

Corollary 4.7. Let $f : [a,b] \to \mathbb{R}$ be monotone. Then, f is absolutely continuous on $[a,b] \iff \int_{(a,b)} f' = f(b) - f(a)$.

Proof. $\implies$ Follows from the theorem.

$\impliedby$ Assume $\int_{(a,b)} f' = f(b) - f(a)$. Let $x \in [a,b]$ and WLOG assume that f is increasing. Then, $\int_{(a,x)} f' \le f(x) - f(a)$ and $\int_{(x,b)} f' \le f(b) - f(x)$.

It follows that $f(x) - f(a) = f(x) - f(b) + f(b) - f(a) \le \int_{(a,b)} f' - \int_{(x,b)} f' \le \int_{(a,x)} f'$ so $\int_{(a,x)} f' = f(x) - f(a)$ which gives that f is absolutely continuous on $[a,b]$. □

Corollary 4.8. Every function of bounded variation $f : [a,b] \to \mathbb{R}$ can be written as $f = f_{abs} + f_{sing}$ where f_{abs} is absolutely continuous and $f'_{sing} = 0$ a.e. in (a,b) (singular part).

Proof. Define $f_{abs}(x) = \int_{(a,x)} f'$ and $f_{sing} = f - f_{abs}$. Then f_{abs} is absolutely continuous and $f'_{sing} = f' - f' = 0$. □

5 Lebesgue measure and integral in $\mathbb{R}^d, d \ge 2$

We define the **outer measure** of $A \subseteq \mathbb{R}^d$ as:

$$m^*(A) = \inf\left\{\sum_{k=1}^{\infty} vol(R_k) : R_k = (a_{k,1}, b_{k,1} \times ... \times (a_{k,d} b_{k,d})) \text{ open bounded rectangles such that} A \subseteq \bigcup_{k \in \mathbb{N}} R_k\right\}$$

where $vol(R_k) = \Pi_{i=1}^{d}(b_{k,i} - a_{k,i})$

Proposition 5.1. Every open set $O \subseteq \mathbb{R}^d$ can be written as $O = \bigcup_{k\in\mathbb{N}} \overline{Q_k}$ where $Q_k = c_k + l_k(-\frac{1}{2}, \frac{1}{2})$ (where $c_k \in \mathbb{R}^d$ is the center of the cube and we add $\frac{1}{2} l_k$ in every direction) are disjoint, open, bounded cubes.

Take a grid of size 1 and fit in all possible cubes in O. Then, repeat with size $\frac{1}{2}$ to get more cubes to cover O and repeat infinitely many time.

The process works since for every point x in O, we can find a cube that contains x and that is contained in O so eventually by this process we capture that point.

Proof. Let $O \subseteq \mathbb{R}^d$ be open. For every $k \in \mathbb{N}$, let C_k be the set of all closed cubes with vertices in $2^{-}k\mathbb{Z}$ (grid).

Let $C_1(O) = \{Q \in C_1 : Q \subseteq O\}$ and $O_1 = \bigcup_{Q \in C_1(O)} Q$

By induction, construct $C_k(O) = \{Q \in C_k : Q \subseteq O \wedge Q \not\subseteq \bigcup_{j=1}^{k=1} O_j\}$ and $O_k = \bigcup_{Q \in C_k(O)} Q$.

Then, we show that $O = \bigcup_{k\in\mathbb{N}} O_k = \bigcup_{k\in\mathbb{N}} \bigcup_{Q\in C_k(O)} Q$.

Since $\forall Q \in C_k(O) : Q \subseteq O$, we have $\bigcup_{k\in\mathbb{N}} O_k \subseteq O$.

Conversely, let $x \in O$. Then, since $2^{-k} \to 0$ as $f \to \infty$ and O is open, $\exists k_0 \in \mathbb{N}$ and a $Q_0 \in C_{k_0}$ such that $x \in Q_0$ and $Q_0 \subseteq O$. From our construction, we must have that $Q_0 \subseteq \bigcup_{k=1}^{k_0} O_k$ (either included at stage k_0 or added at a previous stage). This proves that $x \in \bigcup_{k\in\mathbb{N}} O_k$. □

5.1 Fubini and Tonelli's theorems

Let d_1 and $d_2 \in \mathbb{N}$ be such that $d_1 + d_2 = d$.

For every $E \subseteq \mathbb{R}^d$ and $(x_0, y_0) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d-2} = \mathbb{R}^d$, we denote $E_{x_0} = \{y \in \mathbb{R}^{d_2} : (x_0, y) \in E\}$ and $E_{y_0} = \{x \in \mathbb{R}^{d_1} : (x, y_0) \in E\}$.

$\forall f : E \to \overline{\mathbb{R}}$ we denote $f_{x_0} : E_{x_0} \to \overline{\mathbb{R}}$, $y \to f(x_0, y)$ and $f_{y_0} : E_{y_0} \to \overline{\mathbb{R}}$, $x \to f(x, y_0)$.

Remark. We must be conscious of the following:

1. E measurable does not imply E_x, E_y measurable. Let $E = E_1 \times \{0\}^{d-1}$ where E_1 is not measurable in $[0,1]$. Then, $m^*(E) \leq vol((-\epsilon, 1+\epsilon) \times (\epsilon, \epsilon)^{d-1}) = (1+2\epsilon)2\epsilon^{d-1} \to 0$ so E is measurable ($m(E) = 0$).

2. It is not always true that:

$$\int_{\mathbb{R}^{d_1}} (\int_{\mathbb{R}^{d_2}} f(x,y)dy)dx = \int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x,y)dx)dy$$

   Even if these integrals are well-defined. In fact, consider:

$$\int_{[0,1]} (\int_{[0,1]} \frac{x^2 - y^2}{(x^2+y^2)^2} dy)dx = \int_{[0,1][\frac{y}{x^2+y^2}]_0^1 dx} = \int_{[0,1]} \frac{1}{x^2+1} dx = [arctan(x)]_0^1 = \frac{\pi}{4}$$

   Changing the order of integration gives us the negation of the previous result $\implies = -\frac{\pi}{4}$ and thus they are not equal.

Theorem 5.2. Fubini's Theorem (version in $\mathbb{R}^d$): Let $f : \mathbb{R}^d \to \overline{\mathbb{R}}$ be integrable over $\mathbb{R}^d$. Then:

1. For a.e. $y \in \mathbb{R}^{d_2}$, f_y is integrable over $\mathbb{R}^{d_1}$
2. $y \to \int_{\mathbb{R}^{d_1}} f_y$ is integrable over $\mathbb{R}^{d_2}$.
3. $\int_{\mathbb{R}^{d_2}} (\int_{\mathbb{R}^{d_1}} f(x,y)dx)dy = \int_{\mathbb{R}^d} f$

Remark. The roles of x and y in Fubini's Theorem can be inverted so that in particular

$$\int_{\mathbb{R}^{d_1}} (\int_{\mathbb{R}^{d_2}} f(x,y)dy)dx = \int_{\mathbb{R}^d} f$$

Proof. Let $F = \{f : \mathbb{R}^d \to \overline{\mathbb{R}}$ integrable such that 1,2,3 hold$\}$

Step 1: $\forall \alpha_1..\alpha_n \in \mathbb{R}, f_1...f_n \in F \implies \alpha_1 f_1 + ...\alpha_n f_n \in F$, from the linearity of the integral.

Step 2: Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of functions in F such that:

1. a) $[\forall n \in \mathbb{N} f_n \leq f_{n+1}]$ or $[\forall n \in \mathbb{N} f_n \geq f_{n+1}]$
2. a) $\forall (x,y) \in \mathbb{R}^d : \lim_{n\to\infty} f_n(x,y) = f(x,y)$ (not only a.e.) for some function $f : \mathbb{R}^d \to \overline{\mathbb{R}}$

Then, $f \in F$. WLOG we may assume that $f_{n+1} \geq f_n$ (otherwise consider $-f_n$) and that $f_n \geq 0$ (otherwise just consider $f_n - f_1$).

Since $f_n \in F$, we have that 1,2,3 hold. From 1. and 2.a), we obtain that for a.e. $y \in \mathbb{R}^{d_2}$, f_y is measurable.

By the MCT, we obtain $\lim_{n\to\infty} \int_{\mathbb{R}^{d_1}} f_{n,y} \to \int_{\mathbb{R}^{d_1}} f_y$. By 2, it follows that $y \to \int_{\mathbb{R}^{d_1}} f$ is measurable.

Once again, by the MCT, we obtain that $\lim_{n\to\infty} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} {}_{f_{n,y}} = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} {}_{f_y}$

On the other hand, the MCT also gives that $\lim_{n\to\infty} \int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f$. Passing to the limit in 3., we obtain 3. $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^d} f$.

Since f is integrable in $\mathbb{R}^d$, it follows that $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f_y < \infty$ i.e $y \to \int_{\mathbb{R}^{d_1}} f_y$ is integrable. Hence $\int_{\mathbb{R}^{d_2}} f_y < \infty$ for a.e. $y \in \mathbb{R}^{d_2}$.

Step 3: $\forall E \subseteq \mathbb{R}^d$ measurable.

MISSING REST OF PROOF □

Theorem 5.3. Tonelli's Theorem: Let $f : \mathbb{R}^d \to [0,\infty]$ be measurable, then:

1. For a.e. $y \in \mathbb{R}^{d_2}$, f_Y is increasing measurable on $\mathbb{R}^{d_2}$
2. $y \to \int_{\mathbb{R}^{d_1}} f_x$ is nonnegative measurable on $\mathbb{R}^{d_1}$
3. $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f_y = \int_{\mathbb{R}^d} f$

Remark. Once again, the role of x and y can be exchanged.

Proof. Let,

$$f_n(x) = \begin{cases} \min(f(x), n) & \text{if } f(x) \leq n \\ 0 & \text{if } |x| \geq n \end{cases}$$

$\forall n \in \mathbb{N}$. Since f_n is measurable and $0 \leq f_n \leq n\chi_{B(0,n)}$ so f_n is integrable, we can apply Fubini's Theorem which gives:

1. For a.e. $y \in \mathbb{R}^d$, $(f_n)_y)$ is integrable over $\mathbb{R}^{d_1}$
2. $y \to \int_{\mathbb{R}^{d_1}} (f_n)_y$ is integrable over $\mathbb{R}^{d_2}$
3. $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} (f_n)_y = \int_{\mathbb{R}^d} f_n$

Since $(f_n)_y \to f_y$ pointwise in $\mathbb{R}^d$, from 1' we obtain f is mbl for a.e. $y \in \mathbb{R}^d$. Since $f_{n+1} \geq f_n \geq 0$, the MCT gives $\lim_{n\to\infty} \int_{\mathbb{R}^{d_1}} (f_n)_y = \int_{\mathbb{R}^{d_1}} f_y) \implies 1$.

From 2', it follows that $y \to \int_{\mathbb{R}^{d_1}} f$ is measurable and thus we have 2.

For 3, by the MCT, it follows that:

$$\lim_{n\to\infty} \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} (f_n)_y = \int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} f$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^d} f_n = \int_{\mathbb{R}^d} f$$

then passing to the limit of 3', we obtain 3

□

Corollary 5.4. let $E \subseteq \mathbb{R}^d$ be a measurable set. Then:

1. For a.e. $y \in \mathbb{R}^{d_2}$, E_y is measurable
2. $y \to m(E_y)$ is measurable.
3. $m(E) = \int_{\mathbb{R}^{d_2}} m(E_y)dy$

Proof. Apply Tonelli's Theorem with $f = \chi_E$. Then:

1. ' For a.e. $y \in R^d$, $(\chi_E)_y$ is measurable
2. ' $y \to \int_{\mathbb{R}^{d_1}} (\chi_E)_y$ is measurable.
3. ' $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} (\chi_E)_y \int_{\mathbb{R}} \chi_E = m(E)$

Since $(\chi_E)_y = \chi_{E_y}$ and $E_y = \chi_{E_y}^{-1}(\{1\}) = \chi_{E_y}^{-1}([-\infty,1] \cap \chi_{E_y}^{-1}([1,\infty])$ we obtain that E_y is measurable for a.e. $y \in \mathbb{R}^{d_2}$. This completes 1.

For the other 2, it follows from 2' and 3' by remarking that $m(E_y) = \int_{\mathbb{R}^{d_1}} \chi_E$. □

Corollary 5.5. General version of Tonelli's Theorem: Let $E \subseteq \mathbb{R}^d$ be measurable and $f : E \to [0,\infty]$ be measurable. Then:

1. For a.e. $y \in \mathbb{R}^{d_2}$, f is nonnegative measurable on E_y.
2. $y \to \int_{E_y} f_y$ is nonnegative measurable on $\mathbb{R}^{d_2}$
3. $\int_{\mathbb{R}^{d_2}} \int_{E_y} f_y = \int_E f$

Proof. Apply Tonelli's Theorem to $\tilde{f}(x) = f(x)$ if $x \in E$, 0 otherwise. This new function is measurable since E and f are measurable. Then:

1. ' For a.e. $y \in \mathbb{R}^{d_2}$, $\tilde{f}_y$ is nonnegative measurable on $\mathbb{R}^{d_1}$
2. ' $y \to \int_{\mathbb{R}^{d_2}} \tilde{f}_y$ is nonnegative measurable on $\mathbb{R}^{d_2}$
3. ' $\int_{\mathbb{R}^{d_2}} \int_{\mathbb{R}^{d_1}} \tilde{f}_y = \int_{\mathbb{R}^d} f$

Since $f_y = \tilde{f}_y|_{E_y}$ and $\tilde{f}_y$ and E_y are measurable, we obtain that f_y is measurable on E_y for a.e. $y \in \mathbb{R}^{d_2}$. Then, 2 and 3 follow from 2' and 3' by remarking that $\int_{E_Y} f_Y = \int_{\mathbb{R}^{d_1}} \tilde{f}_y$. □

Corollary 5.6. General version of Fubini's Theorem: Let $E \subseteq \mathbb{R}^d$ be measurable and $f : E \to [-\infty, \infty]$ be integrable. Then:

1. For a.e. $y \in \mathbb{R}^{d_2}$, f_y is integrable on E_y.
2. $y \to \int_{E_y} f_y$ is integrable on $\mathbb{R}^{d_2}$
3. $\int_{\mathbb{R}^{d_2}} \int_{E_y} f_y = \int_E f$

Proof. Apply Fubini's Theorem to $\tilde{f}(x) = f(x)$ if $x \in E$, 0 otherwise. □

Theorem 5.7. Let E_1 and E_2 be measurable sets on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively. Then, $E_1 \times E_2$ is measurable and

$$m(E_1 \times E_2) = \begin{cases} m(E_1)m(E_2) & \text{if } m(E_1) \neq 0, m(E_2) \neq 0 \\ 0 & \text{if either has measure 0} \end{cases}$$

Proof. **Step 1:** Show that $\forall E_1, E_2 \subseteq \mathbb{R}^d$:

$$m^*(E_1 \times E_2) \leq \begin{cases} m^*(E_1)m^*(E_2) & \text{if both have nonzero outer measure} \\ & \text{if one has zero outer measure} \end{cases}$$

WLOG, we may assume that $m^*(E_1) \leq m^*(E_2)$.

Case $m^*(E_2) = \infty$ and $m^*(E_1) > 0$. In this case, $m^*(E_1)m^*(E_2) = \infty$ so there is nothing to prove.

Case $m^*(E_2) < \infty$ and $m^*(E_1) > 0$: Let $\epsilon > 0$ and $(R_k)_{k\in\mathbb{N}}$ and $(R'_k)_{k\in\mathbb{N}}$ be open bounded rectangles that cover E_1 and E_2 respectively with $\sum_{k=1}^{\infty} vol(R_k) < m^*(E_1) + \epsilon$ and $\sum_{k=1}^{\infty} vol(R'_k) < m^*(E_2) + \epsilon$.

Then, $E_1 \times E_2 \subseteq (\bigcup_{k\in\mathbb{N}} R_k) \times (\bigcup_{j\in\mathbb{N}} R'_j) = \bigcup_{k,j\in\mathbb{N}} R_k \times R'_j$. Furthermore, $R_k \times R'_j$ are open bounded rectangles and:

$$\begin{aligned} m^*(E_1 \times E_2) &\leq \sum_{j,k=1}^{\infty} m^*(R_k \times R'_j) = \sum_{k,j=1}^{\infty} vol(R_k \times R'_j) = \sum_{k,j=1}^{\infty} vol(R_k)vol(R'_j) \\ &= \sum_{k=1}^{\infty} vol(R_k) \sum_{j=1}^{\infty} vol(R'_j) < (m^*(E_1) + \epsilon)(m^*(E_2) + \epsilon) \end{aligned}$$

Case $m^*(E_1) = 0$: Write $E_2 = \bigcup_{k\in\mathbb{N}} E_2 \cap [-k,k]^{d_2}$. Then, $E_1 \times E_2 \subseteq \bigcup_{k\in\mathbb{N}} E_1 \times (E_2 \cap [-k,k]^{d_2})$. Then, from the above case:

$$m^*(E_1 \times (E_2 \cap [-k,k]^{d_2})) = m^*(E_1)m^*(E_2 \cap [-k,k]) = 0$$

since the second term is finite and the first is 0. Thus, by subadditivity, it follows that $m^*(E_1 \times E_2) = 0$.

Step 2: Conclusion. Since E_1 and E_2 are measurable, $\exists G_1, G_2$ G_δ sets such that $E_1 \subseteq G_1, E_2 \subseteq G_2$ and $m(G_1 - E_1) = m(G - E_2) = 0$. Then, $G_1 = \bigcap_{n\in\mathbb{N}} O_n, G_2 = \bigcap_{n\in\mathbb{N}} O'_n$ and $G_1 \times G_2 = \bigcap_{n\in\mathbb{N}} O_n \times \bigcap_{k\in\mathbb{N}} O'_k = \bigcap_{k,n\in\mathbb{N}} O_n \times O'_k$.

The sets $O_n \times O'_k$ are open as products of open sets so $G_1 \times G_2$ is a G_δ set in $\mathbb{R}^d$. Moreover, $E_1 \times E_2 \subseteq G_1 \times G_2$ since $E_1 \subseteq G_1$ and $E_2 \subseteq G_2$ and $m^*((G_1 \times G_2) - (E_1 \times E_2)) = m^*([G_1 \times (G_2 - E_2)] \cup [G_2 \times (G_1 - E_1)]) \leq m^*(G_1 \times (G_2 - E_2)) + m^*(G_2 \times (G_1 - E_1)) \leq m^*(G_1)m^*(G_2 - E_2) + m^*(G_2)m^*(G_1 - E_1) = 0$.

Since $E_1 \times E_2$ is measurable. Then, by the first corollary of Tonneli's Theorem, given that $m(E_1 \times E_2) = \int_{E_2} m(E_1) = m(E_1)m(E_2)$ if both are nonzero, 0 otherwise.

Since $(E_1 \times E_2)_y = E_1$ if $y \in E_2$ and $\emptyset$ otherwise. □

Corollary 5.8. Let $E_1 \subseteq \mathbb{R}^{d_1}, E_2 \subseteq \mathbb{R}^{d_2}$ be measurable and $f : E_1 \to \overline{\mathbb{R}}$ be measurable. Then, $\tilde{f} : E_1 \times E_2 \to \overline{\mathbb{R}}, \tilde{f}(x, y) = f(x)$ is measurable.

Proof. $\forall c \in \mathbb{R} : \tilde{f}^{-1}([-\infty, c)) = f^{-1}([-\infty, c]) \times E_2$ where both are measurable. Thus, we can conclude that $\tilde{f}^{-1}([-\infty, c))$ is measurable. □

Theorem 5.9. Assume that $d_1 = d - 1$ and $d_2 = 1$. Let $E_1 \subseteq \mathbb{R}^{d-1}$ be measurable and $f : E_1 \to [0, \infty]$. Then, f is measurable iff the set $A = \{(x, y) \in E_1 \times \mathbb{R} : 0 < x < f(x)\}$ is measurable. Moreover, if f is measurable, then $m(A) = \int_{E_1} f$.

Proof. Assume that f is measurable. Write $A = \{(x, y) \in E_1 \times (0, \infty] : f(x, y) = y - f(x) < 0\} = \tilde{f}^{-1}([-\infty, 0))$ where $\tilde{f} : E_1 \times (0, \infty)$.

From the previous corollary, $(x, y) \to f(x)$ and $(x, y) \to y$ are measurable on $E_1 \times (0, \infty)$ and so $\tilde{f}$ is measurable on $E_1 \times (0, \infty)$. Therefore, A is measurable.

Assume A is measurable. Then by Corollary 1 of Tonelli's Theorem, $x \to m(A_x) = 0$ if $x \notin E_1$ or $f(x)$ otherwise. Then, $G|_{E_1} = f$ is measurable. Furthermore, $m(A) = \int_{\mathbb{R}^{d_1}} m(A_x)dx = \int_{\mathbb{R}^{d_1}} g = \int_{E_1} f$ □

6 Hausdorff measure

Def. For every $s > 0$ and $A \subseteq \mathbb{R}^d$, we define the **s-dimensional Hausdorff exterior measure** of A as the number:

$$m_s^*(A) = \lim_{\delta \to 0} H_s^\delta(A)$$

where

$$H_s^\delta(A) = \inf\left\{\sum_{k=1}^{\infty} diam(A_k)^s : A \subseteq \bigcup_{k\in\mathbb{N}} A_k, diam(A_k) < \delta\right\}$$

and

$$diam(A_k) = \sup\{|x - y| : x, y \in A_k\}$$

Theorem 6.1. $\forall A \subseteq R^d : \exists! d_A \in [0,d]$ such that $\forall s < d_A : m_s^*(A) = \infty$ and $\forall s > d_A m_s^*(A) = 0$. We call d_A the **Hausdorff dimension** of A and we denote $d_A = dim_H(A)$.

Proof. We show that $\forall \alpha < \beta \in [0,d]$:

1. $m_\alpha^*(A) < \infty \implies m_\beta^*(A) = 0$
2. $m_\beta^*(A) > 0 \implies m_\alpha^*(A) = \infty$

It suffices to prove 1. since 2. is the contrapositive.

Assume $\alpha < \beta$ and $m^*(A) < \infty$. Let $(A_k)_{k\in\mathbb{N}}$ be an appropriate cover. Simply observe that $(diamA_k)^\beta = (diamA_k)^{\beta-\alpha}(diamA_k)^\alpha < \delta^{\beta-\alpha}(diamA_k)^\alpha$.

Hence $H_\beta^\delta(A) \le \delta^{\beta-\alpha} H_\alpha^s(A) \to 0$ as $\delta \to 0$.

Let $d_\alpha = \sup\{\alpha \in [0,d] : m_\alpha^*(A) = \infty\}$ or 0 if the set is empty. Then 1. and 2. implies the required properties for d_α. □

Remark.
1. Like for the Lebesgue's exterior measure, we can prove monotonicity, subadditivity and measurability of Borel sets, etc. and construct the associated integral.
2. In case $s = d$, it can be shown that $m_d^*(A) = 2^d m^*(A)$ since $2^d = m^*(B^d)$ where B^d is the unit ball in $\mathbb{R}^d$.
3. In case $s = 0$, it can be shown that $m_0^*(A) = |A|$ (i.e. the **counting measure**).
4. It can be shown that for every Lipschitz function $f : [0,1] \to \mathbb{R}^d$ that is injective except for a finite number of points, the curve $C = f([0,1])$ is of Hausdorff dimension 1.
5. It can be shown that the Cantor set has Hausdorff dimension $\frac{ln2}{ln3}$.

Proof. Half proof of 5. □